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ABSTRACT

This book is a partially annotated bibliography of books, articles and periodicals concerned with mathematical games, puzzles, tricks, amusements, and paradoxes. Volume 2 follows an original monograph which has gone through three editions. The present volume not only brings the literature up to date but also includes material which was omitted in Volume 1. The book is intended for both the professional and amateur mathematician. This guide can serve as a place to look for source materials and will be helpful to students engaged in research. Many non-technical references are included for the layman interested in mathematics as a hobby. One useful improvement over Volume 1 is that the number of subheadings has been more than doubled. (FL)

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A BIBLIOGRAPHY OF recreational mathematics

William L. Schaaf

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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A BIBLIOGRAPHY OF

recreational mathematics

volume 2

Pleaſure with Profit:

Conſiſting of

RECREATIONS

OF

DIVERS KINDS,

VIZ.

Numerical,	*Aſtronomical,*	*Automatical,*
Geometrical,	*Horometrical,*	*Chymical,*
Mechanical,	*Cryptographical,*	and
Statical,	*Magnetical,*	*Hiſtorical.*

Publiſhed to Recreate Ingenious Spirits; and to induce them to make farther ſcrutiny into theſe (and the like) SUBLIME SCIENCES.

AND

To divert them from following ſuch Vices, to which Youth (in this Age) are ſo much Inclin'd.

By WILLIAM LEYBOURN, Philomathes.

To this Work is alſo Annext,

A TREATISE of ALGEBRA,

According to the late Improvements, applied to *Numerical Queſtions* and *Geometry*; with a NEW SERIES for the ſpeedy *Extraction of Roots*; as alſo a CONVERGING SERIES for all manner of *Adfected Equations.*

By R. SAULT, Maſter of the *Mathematick School* in *Adam's* Court, in *Broadſtreet*, near the *Royal Exchange*, *LONDON.*

LONDON:

Printed for **Richard Baldwin**, and **John Dunton**; near the *Oxford-Arms* in *Warwick-Lane*: And at the *Raven* in the *Poultrey*. 1694.

R. Chubb.

A BIBLIOGRAPHY OF

recreational mathematics

VOLUME

2

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Preface

Popular interest in recreational mathematics would seem to be at an all-time high. The last few decades have witnessed the appearance of literally scores of books and hundreds of articles—eloquent testimony to the wide appeal of mathematical games and puzzles, tricks, pastimes, amusements, brainteasers, and paradoxes. In addition to this embarrassment of riches there is to be noted a shift in emphasis in recent years. Whereas formerly much attention was given to numerical curiosities, magic squares, the fourth dimension. tangrams, dissections, angle trisection and circle squaring, contemporary writers tend to devote more attention to such matters as acrostics, alphametics. colored cubes, combinatorics, electronic computers (primes, chess playing, etc.), flexahedrons, game strategy, inferential and logic problems, modular arithmetic, networks and graphs, packing problems, tessellations, polyominoes. probability (queuing, etc.), scales of notation, the Soma cube, topological recreations.

In this connection one should mention some of the outstanding leaders in the field. The close of the nineteenth century and the early decades of the present century saw the work of W. Ahrens, W. W. Rouse Ball, Lewis Carroll. Henry Dudeney, F. Fitting, E. Fourrey, J. Ghersi, Bruno Kerst, W. Lietzmann, Sam Loyd, E. Lucas, L. Mittenzwey, Hubert Phillips, and Hermann Schubert. In more recent years the roster of notable contributors would include, among others, the following: A. R. Amir-Moéz, Irving Adler, Leon Bankoff, Stephen Barr, L. G. Brandes, Maxey Brooke, Mannis Charosh, John Conway, Martyn Cundy, A. P. Domoryad, Martin Gardner, Michael Goldberg, Solomon Golomb, Philip Heaford, Piet Hein, J. A. H. Hunter, Murray Klamkin, Boris Kordemsky, Sidney Kravitz, Harry Lindgren, Joseph Madachy, Karl Menninger, Leo Moser, T. H. O'Beirne, C. Stanley Ogilvy, the late Hans Rademacher, Howard Saar, Sidney Sackson, Fred Schuh, Hugo Steinhaus, Otto Toeplitz, Charles W. Trigg, and William Tutte.

Some indication of the deep interest in recreational mathematics evinced by teachers is the fact that since this bibliography first appeared some fifteen years ago it has passed through three editions and seven printings. Meanwhile. the literature has become so extensive that it was felt a second volume would prove more serviceable than a revision of the original monograph. Thus the present monograph, volume 2, not only brings the literature up to date but also fills in many gaps and omissions. It contains none of the earlier material except

the supplementary periodical references, which occupied the last eight pages of the third edition. These references have been incorporated in the new classification in the present volume.

The organization of the contents of volume 2 is such that the major topics parallel the original classification fairly closely, making it relatively easy to consult both volumes on a given topic. There are several innovations which it is hoped will materially increase the usefulness of the monograph. To begin with, the number of subheadings has been more than doubled—and this should help the reader to track down a particular subject or special type of recreation. Secondly, in addition to periodical references, many references to "specialized" books are now included—books dealing solely with a single topic, such as mazes, or dissections, or tangrams, or inferential problems. Finally, there are now included also many references to specific topics in important general works on recreational mathematics, such as books by Ball, Domoryad, Gardner, Madachy, O'Beirne, Schuh, and others.

As in the case of volume 1, the classification of mathematical recreations is at best an arbitrary affair, reflecting the bibliographer's personal judgment. I have again endeavored to make the coverage sufficiently comprehensive to include comparatively simple as well as highly sophisticated material—topics usually thought of as mathematical puzzles and pastimes as well as certain fringe topics such as mathematical humor, mathematics in sports, mathematics in art, literature, and music, and mathematics in nature. In this way it is hoped to reach a wide spectrum of readers: students, teachers, laymen, amateurs, and professional mathematicians.

I take this opportunity to acknowledge my indebtedness over the past years to many friends and colleagues, including, among others, Professor H. S. M. Coxeter, Mr. J. A. H. Hunter, Mr. Joseph S. Madachy, and Professor Adrian Struyk. I am indebted also to Mr. Leon Bankoff for data on the arbelos and other geometric recreations; to Maxey Brooke for help in connection with magic squares, Möbius bands, and other topological ideas; to Professor R. L. Morton in connection with perfect numbers and amicable numbers; to Mr. Charles W. Trigg for assistance with linkages, packing problems, and tessellations. My greatest debt is to Martin Gardner, not only for specific help, but for encouragement and inspiration. I am grateful also to the several publishers from whose works brief excerpts have been quoted; to the National Council of Teachers of Mathematics for their generous cooperation; and to my wife for her infinite patience in many ways.

William L. Schaaf

Boca Raton, Florida
September 1969

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"Have you guessed the riddle yet?" the Hatter said, turning to Alice again.

"No, I give it up," Alice replied. "What's the answer?"

"I haven't the slightest idea," said the Hatter.

"Nor I," said the March Hare.

Alice sighed wearily. "I think you might do something better with the time," she said, "than wasting it in asking riddles that have no answers."

LEWIS CARROLL
Alice's Adventures in Wonderland (1865)

Principal Abbreviations Used

Am.M.Mo. = *American Mathematical Monthly*
A.T. = *Arithmetic Teacher*
Fib.Q. = *Fibonacci Quarterly*
J.R.M. = *Journal of Recreational Mathematics*
M.Gaz. = *Mathematical Gazette*
M.Mag. = *Mathematics Magazine*
M.T. = *Mathematics Teacher*
Math.Tchg. = *Mathematics Teaching* (England)
N.M.M. = *National Mathematics Magazine*
NCTM = National Council of Teachers of Mathematics
Rec.M.M. = *Recreational Mathematics Magazine*
Sci.Am. = *Scientific American*
Sci.Mo. = *Scientific Monthly*
Scrip.M. = *Scripta Mathematica*
S.S.M. = *School Science and Mathematics*
Z.M.N.U. = *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*

Chapter 1

Algebraic and Arithmetical Recreations

1.1 The Abacus

Though not ordinarily thought of as a mathematical "recreation," use of the abacus has considerable recreational value. Moreover, since contemporary school mathematics has given increased emphasis to numeration systems and number bases, familiarity with the abacus has acquired a new significance. Common forms of the abacus include the Chinese *suan-pan*, the Japanese *soroban*, and the Russian *s'choty*. Competitive contests from time to time between experienced desk-calculator operators and skilled users of the abacus generally evoke much interest, particularly as the abacus often comes off very creditably.

Flewelling, R. W. Abacus and our ancestors. *A.T.* 7:104–6; Feb. 1960.

Japan Chamber of Commerce and Industry. *Soroban, the Japanese Abacus: Its Use and Practice.* Rutland, Vt.: Charles E. Tuttle Co., 1967. 96 pp.

Kojima, Takashi. *Advanced Abacus: Japanese Theory and Practice.* Rutland, Vt.: Charles E. Tuttle Co., 1964. 159 pp.

A sequel to the author's *The Japanese Abacus: Its Use and Theory.*

Li, S. T. Origin and development of the Chinese abacus. *Journal, Association of Computing Machinery* 6:102–10; 1959.

Yoshino, Y. *The Japanese Abacus Explained.* (Introd. by Martin Gardner.) 2d ed. New York: Dover Publications, 1963. 240 pp.

1.2 Acrostics; Crossword Puzzles

Acrostics, anagrams, and crossword puzzles are so familiar that no explanation would seem called for except to point out that they frequently appear with mathematical words, phrases, or sentences. A palindrome is a word, a phrase, a sentence, or a number which reads the same from right to

left as from left to right. Perhaps the most widely known palindromic sentence is "ABLE WAS I ERE I SAW ELBA."

Anne Agnes von Steiger, Sister. A Christmas puzzle. *M.T.* 60:848–49; Dec. 1967.
A geometry-oriented dot-joining picture puzzle.

Anning, Norman. Fun with palindromes. *Scrip.M.* 22:227; 1956.

Bishop, David. A mathematical diversion. *M.T.* 58:527; Oct. 1965.
Similar to a crossword puzzle.

DeJong, L. Mathematics crossword. *S.S.M.* 62:45–46; Jan. 1962.

Eckler, A. Ross. Word groups with mathematical structure. *Word Ways*, vol. 1, no. 4; Nov. 1968.

Friend, J. Newton. *Numbers: Fun and Facts.* New York: Charles Scribner's Sons, 1954.
Palindromes: chap. 8, pp. 101–10.

Gardner, Martin. Dr. Matrix: A talk on acrostics. *Sci.Am.* 216:118–23; Jan. 1967.

Gardner, Martin. Double acrostics. *Sci.Am.* 217:268–76; Sept. 1967.

Hawthorne, Frank, and Cohen, D. I. A. Able was I ere I saw Elba. *Am.M.Mo.* 71:318; Mar. 1964.

Hillman, T. P., and Sirois, Barbara. In the name of geometry. *M.T.* 61:264–65; 267; Mar. 1968.

Jones, Lawrence E. Merry Christmas—Happy New Year. *S.S.M.* 67:766–71; Dec. 1967.

Knox, R. A. *A Book of Acrostics.* London, 1924. 136 pp.
Preface includes history and origin of acrostic puzzles.

Landau, Remy. Permutacrostic. *Rec.M.M.*, no. 11, p. 10; Oct. 1962.

Lindon, J. A. Word cubes and 4-D hypercubes. *Rec.M.M.*, no. 5, pp. 46–49; Oct. 1961.

Lindon, J. A. Word shifts. *Rec.M.M.*, no. 12, pp. 33–40; Dec. 1962.

Moskowitz, Sheila. Mathematics crossword puzzle. *S.S.M.* 65:3–11; Jan. 1965.

Nichols, Thomas. Intersection and union word puzzle. *M.T.* 58:28–29; Jan. 1965.

Nygaard, P. H. Can you solve a dictoform? *S.S.M.* 49:6–8; 1949.

Purdy, C. R. Crossword puzzle on percentage. *M.T.* 33:135; Mar. 1940.

Reys, Robert. The 101 mathematics word search. *M.T.* 60:359, 380; Apr. 1967; 61:500, 507; May 1968.

Trigg, Charles W. A holiday message in a permutacrostic. *S.S.M.* 62:687, 693; Dec. 1962.

Trigg, Charles W. Holiday greeting permutacrostic. *Rec.M.M.*, no. 6, p. 26; Dec. 1961.

Trigg, Charles W. A holiday permutacrostic. *S.S.M.* 66:704; Nov. 1966.

Trigg, Charles W. Mathematical word rebuses. *Rec.M.M.*, no. 10, pp. 16–17; Aug. 1962.

Trigg, Charles W. Mathematical permutacrostic. *Rec.M.M.*, no. 3, p. 22; June 1961.

Trigg, Charles W. Palindromes by addition. *M.Mag.* 40:26–28; Jan. 1967.

1.3 Algebraic Puzzles and Pastimes

"A *puzzle* will mean any problem whose method of solution is not—we will hope—immediate or obvious: and a *paradox* will mean something whose truth and explanation can ordinarily be established only in the face of some initial sales resistance."—T. H. O'Beirne, *Puzzles and Paradoxes.*

Allard, J. Note on squares and cubes. *M.Mag.* 37:210–14; 1964.
El-Milick-type equations and curves.

Collins, K. S. Algebra from a cube. *Math.Tchg.*, no. 46, pp. 58–62; Spring 1969.

Dunn, Angela. *Mathematical Bafflers.* New York: McGraw-Hill Book Co., 1964. 217 pp.

Haga, Enoch. Square-off at squares and cubes. *S.S.M.* 60:122–26; 1960.
Shortcuts to determine squares and cubes, square roots and cube roots.

Jorgenson, Paul S. Fun with graphs. *M.T.* 50:524–25; 1957.
Graphing exercises which result in pictures.

Larsen, Harold, and Saar, Howard. One little, two little . . . . *Rec.M.M.*, no. 8, pp. 37–39; Apr. 1962.
Take-off on verbal problems in the mathematics classroom.

Matthews, Geoffrey. The inequality of the arithmetic and geometric means. *Scrip.M.* 22:233; 1956.

Ogilvy, C. Stanley. Geometric algebra. *Rec.M.M.*, no. 3, pp. 37–39; June 1961.

1.4 Alphametics and Cryptarithms

A cryptarithm is a mathematical problem calling for addition, subtraction, multiplication, or division in which the digits have been replaced by letters of the alphabet or some other symbols. Alphametics, a term invented by J. A. H. Hunter, refers specifically to those cryptarithms in which the combinations of letters make sense. One of the oldest and probably best known of all alphametics is the celebrated

$$\begin{array}{r} \text{SEND} \\ \text{MORE} \\ \hline \text{MONEY} \end{array}$$

Alphametics. *Rec.M.M.*, no. 10, p. 11; Aug. 1962.
Dudeney's classic alphametics.

Austin, A. K. With a dash of mathematics. *Math.Tchg.*, no. 36, pp. 28–30; Autumn 1966.
A note on cryptarithms.

Brooke, Maxey. A Christmas cryptarithm. *M.Mag.* 41:160–61; May 1968.

Brooke, Maxey. *150 Puzzles in Crypt-Arithmetic.* New York: Dover Publications, 1963. 72 pp. (Paper)

Demir, Huseyin. A memorial cryptarithm. *M.Mag.* 38:242; Sept. 1965.

Demir, H., and Joy, Sister Mary. A conditional alphametic. [No. 544]. *M.Mag.* 37:354–55; 1964.

Hunter, J. A. H. An alphametic—and another alphametic. *J.R.M.* 1:35–36; Jan. 1968.

Hunter, J. A. H. Alphametics. *J.R.M.* 1:45–48; Jan. 1968.

Konhauser, J. D. E. Crypta-equivalence. *M.Mag.* 39:248–49; Sept. 1966.

Kravitz, Sidney. The art of solving multiplication type alphametics. *Rec.M.M.*, no. 2, pp. 9–16; Apr. 1961.

Kravitz, Sidney. Some novel cryptarithms. *J.R.M.* 1:237; Oct. 1968.

Kravitz, Sidney, et al. The whispered clue. *M.Mag.* 41:43–44; Jan. 1968.
A cryptarithm with an unusual twist.

Madachy, Joseph. *Mathematics on Vacation.* New York: Charles Scribner's Sons, 1966.
"Alphametics," pp. 178–200.

Problem #179. (Janes, Lebbert, and Simmons). *Pentagon* 24:104; Spring 1965.

Rosenfeld, Azriel. Problem E1681 (Cryptarithm). *Am.M.Mo.* 71:429; 1964.

Sutcliffe, Alan. On setting alphametics. *Rec.M.M.*, no. 10, pp. 8–10; Aug. 1962.

Trigg, Charles W. A vexatious cryptarithm. *M.Mag.* 39:307–8; Nov. 1966.
Discussion of "Vexing = Math," and the impact of computers on the solution of puzzle problems.

Trigg, Charles W. "Crystal" alphametic. (#551). *M.Mag.* 37:196; May 1964.

1.5 Calendar Problems

Many questions concerning the calendar seem to intrigue people: the relation of the Gregorian calendar to the Julian calendar; the determination of leap years; methods for determining the date of Easter; the construction of a perpetual calendar; calendar reform and the thirteen-month World calendar. In addition, interesting sidelights appear, such as the mathematical basis for

the assertion that the thirteenth day of the month is more likely to be a Friday than to be any other day of the week.

Allen, Richard K. Four thousand years of Easter. *Rec.M.M.*, no. 11, pp. 9–10; Oct. 1962.

Bailey, William T. Friday-the-thirteenth. *M.T.* 62:363–64; May 1969.

Bakst, Aaron. *Mathematical Puzzles and Pastimes.* D. Van Nostrand Co., 1965. 206 pp.

Chapter 8, "Harnessing Father Time," deals with the perpetual calendar and similar problems.

A Calendar Square Trick. *Pentagon* 18:49–50; Fall 1958.

Case, John J. Seasoning for the calendar. *Science* 122:648; October 7, 1955.

Ionides, S. A., and Ionides, M. L. *One Day Telleth Another.* London: Arnold, 1939.

Kessler, Donald. How to use the perpetual calendar. *M.T.* 52:555–56; Nov. 1959.

Kravitz, Sidney. The Christian, Mohammedan and Jewish calendars. *Rec.M.M.*, no. 4, pp. 22–25; Aug. 1961.

Leo, Reverend Brother. A mental calendar. *M.T.* 50:438–39; 1957.

Lucas, Édouard. *Récréations Mathématiques.* vol. 4. Paris, 1891.

Le calendrier perpétuel et le calcul automatique des résidues, pp. 3–20.

Morton, R. L. The calendar and modular arithmetic. *Mathematics Student Journal,* vol. 9, no. 4, pp. 1–3; May 1962.

O'Beirne, T. H. *Puzzles and Paradoxes.* New York: Oxford University Press, 1965. 238 pp.

Chapter 10: "Ten Divisions Lead to Easter," pp. 168–84.

Oliver, S. Ron. Note on a calendar problem. *Pentagon* 28:49–50; Fall 1968.

Perpetual calendar. *M.T.* 45:554; Nov. 1952.

Primrose, E. J. F. The mathematics of Easter. *M.Gaz.* 35:225–27; 1951.

Simon, William. *Mathematical Magic.* Charles Scribner's Sons, 1964.

Calendar Magic: pp. 60–80.

Smiley, M. F. When is Easter? *M.T.* 50:310; Apr. 1957.

Strader, W. W. *Five Little Stories.* Washington, D.C.: NCTM, 1960. 16 pp. (Paper)

An unbelievable month of September.

Thornton, Glenn W. The calendar. *Pentagon* 17:10–15; 1957.

To Find Easter "for ever." *Nature,* vol. 13, no. 338, p. 485; Apr. 20, 1876.

Walker, George W. On the rule for leap year. *Science* 123:25; Jan. 6, 1956.

Wylie, C. C. On the rule for leap year. *Science,* vol. 123; Jan. 16, 1956.

1.6 Continued Fractions

Continued fractions were of considerable interest to seventeenth- and eighteenth-century mathematicians. Contemporary mathematicians are even more concerned with continued fractions, particularly in connection with the theory of numbers. Many aspects of mathematics involve or are related to continued fractions, among them the expansion of both rational and irrational numbers, the expansions of π and e, the expansion of logarithmic and trigonometric series, and the solution of Diophantine equations.

Chrystal, G. *Algebra,* vol. 2. Edinburgh, 1889; New York: Chelsea Publishing Co., 1959.

Continued fractions: chapters 32, 34.

Davenport, H. *The Higher Arithmetic.* London: Hutchinson's University Library, 1952; New York: Harper & Row, Torchbooks, 1960.

Continued fractions, pp. 79–114.

Davis, C. S. *Journal, London Mathematical Society* 20:194–98; 1945.

Proofs for continued fractions for e, etc.

Hardy, G. H., and Wright, E. M. *The Theory of Numbers.* 3d ed. Oxford, 1954. Chapter 10.

Niven, Ivan. *Irrational Numbers.* Carus Mathematical Monographs, no. 11. New York: Mathematical Association of America and John Wiley & Sons, 1956.

Continued fractions: pp. 51–67.

Olds, C. D. *Continued Fractions.* New Mathematical Library. New York: Random House, L. W. Singer Co., 1963. 162 pp. (Paper)

Perron, Oskar. *Die Lehre von den Kettenbrüchen.* Leipzig and Berlin: Teubner, 1929.

Wall, H. S. *Analytic Theory of Continued Fractions.* Princeton, N.J.: D. Van Nostrand Co., 1948.

1.7 Cross-Number Puzzles

Cross-number puzzles, analagous to crossword puzzles, have enjoyed considerable popularity in recent years, not only as a diversion from routine schoolwork, but also, in the earlier grades, both as motivation and as a device for practice.

Alfred, Brother. Cross number puzzle. *Rec.M.M.*, no. 4, p. 13; Aug. 1961.

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1.9 Farey Sequences

If all the proper fractions (written in lowest terms) having denominators not greater than a given number are arranged in order of magnitude, then each fraction in the sequence is equal to the fraction whose numerator is the sum of the two numerators on either side of it, and whose denominator is the sum of the corresponding denominators. Such a sequence is called a Farey sequence. For example: if we choose $n = 3$, we have

$$F_3: \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}.$$

$$\text{For } F_4: \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}.$$

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1.10 Figurate Numbers—Polygonal Numbers

Numbers which can be related to geometric forms were of considerable interest to ancient Greek mathematicians, who were familiar with at least triangular, square, and pentagonal numbers. Thus triangular numbers,

are of the form $\sum_{1}^{n} (n) = \frac{1}{2} n(n+1).$

Such figurate numbers, also known as polygonal numbers, appeared in fifteenth-century arithmetic books, and were probably known to the Chinese as early as about the beginning of the Christian era.

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1.11 Four Fours

This pastime consists of writing consecutive integers using only four 4s, combining them by using any mathematical operation, provided that exactly four 4s and no other digits are used. For example:

$$1 = \frac{4+4-4}{4};\ 2 = \frac{4 \cdot 4}{4+4};\ 3 = \frac{4+4+4}{4};\ \text{etc.}$$

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1.12 Number Bases—Binary Recreations

A system of numeration refers solely to the manner in which numbers are written or expressed by means of symbols. Thus a numeration system consists essentially of an arbitrary constant base number and a set of symbols (digits), with or without a zero, with or without the idea of place value, and with or without the idea of cipherization. To be sure, a numeration system embracing all of these characteristics, such as our modern base-ten Hindu-Arabic system, is highly efficient for most purposes. But for some purposes, bases other than ten prove to have advantages. Thus systems in base two and in base eight are useful in connection with electronic digital computers. Binary and ternary systems are related to certain games and recreations such as Nim and weight puzzles.

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1.13 Number Giants

Who never tried, as a youngster, to write or say "the largest number there is"? In these days of government budgets running to nearly 200 billions of dollars, "large numbers" are no longer unfamiliar. How large is large? An expert bank teller, working forty hours a week, needs more than fifty years to count out one billion one-dollar bills. The first million days of the Christian era will not be reached until the latter part of the year 2739. A light-year comprises about six million million miles (6×10^{12}).

Some well-known number giants include the following:

(*a*) the googol, or 10^{100};

(*b*) the googolplex, or $10^{(10^{100})}$;

(*c*) Skewes's number, $10^{10^{10^{34}}}$.

Would you like to try to imagine the number of zeros in a $\text{googol}^{(\text{googol})}$, or in a $\text{googolplex}^{(\text{googolplex})}$?

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1.14 Number Mysticism and Numerology

Numerology is doubtless as old as astrology, which is to say that since time immemorial men have succumbed to mystery and superstition. In ancient times, certain numbers acquired distinctive "personalities" of their own. The Pythagorean Brotherhood professed faith that numbers could explain all things. The *Gematria* of the Greeks and the Hebrews, the teachings of the Gnostics during the early centuries of Christianity, and medieval number philosophy all testify to the sustained mysticism attributed to numbers. Today this seems to be continued in the still widespread belief in foretelling the future and in reading character by means of numbers.

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1.15 Number Pleasantries and Curiosities

These refer to a wide variety of properties, relations, and oddities among numbers, including identities with the same digits on both sides, telescoped identities, digital invariants, cyclic numbers, invariant sums and powers, multigrades, etc

An *automorphic* number is an integer whose square ends with the given integer; e.g., $(76)^2 = 5{,}776$. A *narcissistic* number is an integer that can somehow be represented by mathematically manipulating its digits; e.g., $153 = 1^3 + 5^3 + 3^3$. *Strobogrammatic* numbers are numbers that read the same after a 180° turn; e.g., 69, 96, and 1001.

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Chapter 2

Number Theory as Recreation

2.1 Amicable Numbers

Two numbers are said to be amicable if they have the property that the sum of the aliquot divisors of either number equals the other number, as for example 220 and 284, a pair probably known to the early Pythagoreans, and for many centuries the only known pair. By 1750 mathematicians had succeeded in identifying some sixty such pairs. The second smallest pair is 1184 and 1210. There are a number of rules for finding pairs of amicable numbers.

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2.2 Diophantine Puzzles

These include problems which lead to mathematical equations whose solutions must be integers. A Diophantine equation is a rational integral alge-

braic equation in which the coefficients of the variables and the absolute term are integers, and for which the solutions must also be in integers. For example, the linear Diophantine equation $ax + by = n$ has a solution if and only if the greatest common divisor of a and b divides n. The theory of numbers provides systematic methods for solving a linear Diophantine equation, even if it contains more than two variables; also for solving simultaneous linear Diophantine equations, as well as nonlinear Diophantine equations.

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2.3 Fermat Numbers

These are numbers of the form $2^{2^n} + 1$, where n is a positive integer. The first four Fermat numbers are thus 3, 5, 17, and 257. Fermat believed that numbers of this form were always prime, but Euler found that the sixth Fermat

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2.4 Fermat Theorems

It was Fermat's habit to write notes in the margins of his books. In one of them he wrote: "However, it is impossible to write a cube as the sum of two cubes, a fourth power as the sum of two fourth powers, and in general, any power beyond the second as the sum of two similar powers. For this I have discovered a truly wonderful proof, but the margin is too small to contain it."

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2.5 Fibonacci Numbers

The Fibonacci sequence of numbers, originating in a thirteenth-century curiosity concerning the propagation of rabbits, is one of the most fascinating aspects of number theory and geometry. The properties and ramifications of the Fibonacci sequence and the related Lucas sequence are amazing and seemingly endless. They are related to the characteristic ratio of the Golden Section, $\phi = (1 + \sqrt{5})/2$. For example:

Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, . . .
Lucas sequence: 1, 3, 4, 7, 11, 18, 29, . . .

$$\phi = (\sqrt{5}+1)/2 \qquad \phi^4 = (3\sqrt{5}+7)/2$$
$$\phi^2 = (\sqrt{5}+3)/2 \qquad \phi^5 = (5\sqrt{5}+11)/2$$
$$\phi^3 = (2\sqrt{5}+4)/2 \qquad \phi^6 = (8\sqrt{5}+18)/2$$

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A number of the form $M_p = 2^p - 1$, where p is prime, is called a Mersenne number. Some Mersenne numbers are composite, others are prime numbers. As of 1963, the existence of twenty-three Mersenne primes was known, e.g., for $p = 2, 3, 5, 7, 13, 19, 31, \ldots 9{,}689, 9{,}941$, and $11{,}213$. The three largest were discovered by the use of an Illiac II electronic computer in a matter of a few hours.

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2.7 Number Theory: General

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"Euclid's formula for finding a perfect number yields only *even* perfect numbers. It is not known whether there exist any odd perfect numbers. Although none has ever been found, mathematicians have not succeeded in proving that none exist. If an odd perfect number does exist, it has been

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2.9 Perfect Numbers

A perfect number is an integer which is equal to the sum of its aliquot divisors, including unity but excluding the number itself. Most numbers are either "abundant" or "deficient." For example: 18 is abundant, its divisors giving a total of 21; 35 is deficient, giving a sum of 13; whereas 6 is perfect, since $1 + 2 + 3 = 6$. The next two perfect numbers are 28 and 496.

Euler's formula for finding even perfect numbers, which was known to Euclid, but not proved until two thousand years later, is given by

$$N = (2^{n-1})\ (2^n - 1).$$

where n is any integer > 1 which makes the second factor, $2^n - 1$, a prime.

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2.10 Prime Numbers

Despite the remarkable advances made in the theory of numbers during the past seventy-five years, many problems still remain unsolved. Thus no method has been found as yet for naming the next prime after any given prime. Likewise, no formula has been developed for finding even one prime greater than a given prime. It has been known since Euclid's time that the number of primes is infinite. And since 1900 it has been proved that the number of primes not exceeding x, called $\pi(x)$, is given by the formula

$$\lim_{x \to \infty} \left(\frac{\pi(x)}{x/\log x} \right) = 1.$$

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2.11 Tests for Divisibility

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Chapter 3

Geometric Recreations

3.1 Apollonian Problem

One of the most celebrated geometric problems of antiquity required to construct the circle or circles tangent (internally or externally) to three given circles. Depending upon the given configuration, there may be as many as eight circles satifying the conditions of the problem, or there may be none.

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3.2 The Arbelos

The *arbelos* of Archimedes, known also as the "shoemaker's knife" which it somewhat resembles, is a plane figure bounded by three semicircles whose centers lie on the same straight line segment, such that the sum of the diameters of the two smaller semicircles (lying on the same side of the segment) equals the diameter of the larger semicircle. Upon close study, the configuration reveals an amazing number of unexpected properties.

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3.3 The Butterfly Problem

Let C be the midpoint of any arbitrary chord $\overline{AB}$ of a given circle; then if $\overline{DE}$ and $\overline{FG}$ are any two chords through C, then $\overline{CH} = \overline{HI}$, where H is the

intersection of $\overline{AB}$ and $\overline{GD}$, and I is the intersection of $\overline{AB}$ and $\overline{EF}$; also, if $\overline{EG}$ meets $\overline{AB}$ extended in J, and $\overline{DF}$ meets $\overline{AB}$ extended in K, then $\overline{CJ} = \overline{CK}$. This so-called "butterfly" property can be shown to hold for ellipses and ovals as well as for circles.

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3.4 The Crossed Ladders

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3.5 Curves and Their Construction

"Mathematicians have a habit of studying, just for the fun of it, things that seem utterly useless; then centuries later their studies turn out to have enormous scientific value. There is no better example of this than the work

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3.6 Curves of Constant Width

When a heavy load, say a small steel safe, is moved horizontally on cylindrical rollers, the load moves parallel to the ground. This is because the cross-section of an ordinary cylindrical roller is a circle, and the distance between two parallel tangents to a circle is constant. A circle is thus a curve of constant width; the width of a circle is the perpendicular distance between parallel tangents.

Curiously, the circle is not the only curve of constant width. In fact, there are infinitely many curves of constant width. One of the best known and simplest is the so-called Reuleaux triangle, formed by three circular arcs with

centers at the vertices of an equilateral triangle whose side equals their radii.

Such curves find useful applications in mechanics and engineering, as, for example, in the design of drills for square holes, or in the design of cams to produce halted motion in feed mechanisms, and similar devices.

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3.7 Curve-stitching

The relatively simple but fascinating art of curve-stitching appears to have been spawned by a delightful little booklet by Edith I. Somervell, entitled *A Rhythmic Approach to Mathematics* (London, 1906). Many designs, some simple, others ornate, but all consisting of straight lines made with colorful threads, are stitched on cards according to some preassigned pattern of punched holes. Even young children are fascinated by designs that they can easily execute. In an introduction to this stimulating booklet, Mrs. Mary

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3.8 Dissection Problems

In general, geometric dissection problems are concerned with cutting certain geometric figures into other desired figures. It can be shown that any rectilinear plane figure can be dissected into any other of the same area by cutting it into a finite number of pieces. As a mathematical recreation, one often wishes to find how to dissect one figure into another figure by dividing it into the *least* possible number of pieces. It is usually impossible to prove that the minimum number of pieces has been determined, and this is doubtless one reason why the subject is far from exhausted. Of course, there are many other kinds of dissections beside minimal dissections, such as dissecting a given square into unequal squares, or into acute triangles; dissecting a cube or a pyramid; Pythagorean dissections; etc.

On the whole, dissection problems offer little by way of practical applications, nor do they involve abstruse mathematical ideas. Indeed, their solution calls largely for empirical and experimental methods, and challenges the ingenuity of amateur and seasoned problem-solvers alike.

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3.9 Finite Geometries

A "miniature" or finite geometry is a system of geometry in which the undefined terms and the postulates are such that the system has only a finite number of points and a finite number of lines. As might be expected, the terms "point" and "line" assume meanings somewhat different from the ordinary meaning or imagery associated with those terms. For example, we speak of a three-point geometry (which is trivial), or a six-point, or a 25-point, or a 31-point geometry. Such geometries can be developed analytically, which reveals an interesting connection between geometry and modular congruences and Galois fields; or they can be developed from the point of view of geometric transformations. In any event, finite geometries shed considerable light on the structure of mathematical systems and on the concept of isomorphism.

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3.10 The Fourth Dimension

"Perhaps here it may be asked why anyone should be interested in the fourth dimension at all. . . . Yet its contemplation gives hints of solution of some of the most absorbing problems of mankind."—Charles W. R. Hooker, *What Is the Fourth Dimension?*

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3.11 Geometric Constructions

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3.13 Linkages

"The conversion of the easily attained circular motion into motion along a straight-line is of prime importance to the engineer and the mechanic. . . . The generation of line motion was no doubt of concern to mathematicians from the time of Archimedes and, because no solution was apparent, many confused this problem with that of squaring the circle. A solution was first given by Sarrus in 1853 and another by Peaucellier in 1864, both of which lay unnoticed until Lipkin, a student of Tschebyschef, independently re-created Peaucellier's mechanism."—Robert C. Yates, in *Multi-Sensory Aids in the Teaching of Mathematics* (18th yearbook of the NCTM).

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3.14 Lissajous Figures

Lissajous figures are interesting and often beautiful curves resulting from the interaction of two harmonic motions at right angles. They can be produced mechanically by means of swinging pendulums (harmonograph) or by means of differential rotating gear wheels (spirograph); also electrically (oscillogram).

A set of curves in rectangular coordinates may be given by a system such as

$$x = g(t),$$
$$y = h(t),$$

where t is a parameter. In general, a typical Lissajous curve is represented by

$$x = a \sin kt$$
$$y = b \sin m(t + \alpha).$$

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3.15 Malfatti's Problem

Malfatti in 1803 proposed the problem of determining the sizes of three nonoverlapping circles of the greatest combined area which could be cut from a given triangle. He believed that the solution consisted of the three circles which are externally tangent to each other, and each of which is tangent to two sides of the triangles. These circles are called the Malfatti circles, and it is now known that the solution is *never* the set of Malfatti circles.

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3.16 Mascheroni Constructions

The geometry of compasses was developed independently by Georg Mohr in Denmark in his book *Euclides Danicus* (1672) and by Lorenzo Mascheroni in Italy in his *Geometria del Compasso* (1797). It is pertinent to distinguish between the "Euclidean compasses" (a collapsible instrument) on the one hand, and the modern compasses (with a set radius) that can not only describe a circle (as with the Euclidean compasses) but can also transfer a distance from one location to another, an operation properly executed by the dividers. As it turns out, every operation that can be performed with the straightedge and dividers can be performed with the straightedge and Euclidean compasses. However, the converse is not true: the straightedge and dividers can do more than the straightedge alone, but not as much as the straightedge and Euclidean compasses.

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3.17 Morley's Theorem

Scarcely any geometric relationship could be more simply stated: The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle. The theorem was first discovered in 1899 by Frank Morley, whose son Christopher wrote *Thunder on the Left* and *Plum Pudding*, among other works. Characterized by Coxeter as "one of the most surprising theorems in elementary geometry," Morley told it to his friends and it spread among mathematicians by word of mouth. Fifteen years later, a simple proof was given by W. E. Philip; since then, many other proofs have been given.

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3.18 Optical Illusions

Various types of optical illusions are well known: for example, illusions created by angles; equivocal figures, often involving shading; depth and distance illusions; deceptions due to misuse of perspective; illusions caused by contour and contrast; illusions involving color, chromatic abberation, and after-images; etc. Many of these involve mathematical concepts and geometric properties; all of them involve physiological or psychological considerations.

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3.19 Squaring the Square—Squared Rectangles

The problem of subdividing a square into smaller squares, no two of which are alike, is a special kind of dissection problem long believed to be unsolvable. Today, by means of electrical-network theory, the "square has been squared."

Along the same lines, we note that a *squared rectangle* is a rectangle that can be dissected into a finite number (two or more) of squares. If no two of these squares have the same size, the squared rectangle is said to be *perfect.* The *order* of a squared rectangle is the number of constituent squares. It is known that there are just two perfect rectangles of order 9, and none of order less than 9.

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3.20 Symmetry

"The pedagogical difficulty of comparing the isosceles triangle *ABC* with itself is sometimes avoided by joining the apex *A* to *D*, the midpoint of the base *BC*. The median *AD* may be regarded as a *mirror* reflecting *B* into *C*.

Accordingly, we say that an isosceles triangle is symmetrical by reflection, or that it has bilateral symmetry."—H. S. M. Coxeter, *Introduction to Geometry.*

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3.21 Tangrams

The formation of designs by means of seven pieces or tiles cut from a square, viz., a smaller square, a rhombus, and five triangles, is an Oriental pastime of Chinese origin, possibly very old. A Greco-Roman version—the stomachion, or loculus of Archimedes—consisted of fourteen pieces cut from a rectangle twice as long as it is wide. The variety of figures possible seems endless, although no mathematical principles are involved.

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Chapter 4

Topological Recreations

4.1 Braids, Knots, and String Figures

Technically speaking, a knot is a set of points in space which is topologically equivalent to a circle. More informally, one can say that a knot is a curve in space formed by looping and interlacing a piece of string in any manner whatever and then joining the ends together. Although any two knots are topologically equivalent, it is not always possible to deform one knot into the other without breaking the string.

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4.3 Graph Theory; Networks

The word "graph" has two distinct meanings in mathematics. On the one hand, a graph is a "curve" which represents a functional relation or some other relation between two or more variables. In the other sense, as used here, a graph, which may be described as a network, is simply a geometrical figure consisting of a number of points and lines connecting some of these points. Originally such linear graphs were associated chiefly with puzzles, but today the theory of graphs and networks finds a wide variety of applications in molecular physics, electrical circuitry, transportation problems, and, in general, in fields such as biology, psychology and economics.

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4.4 Hamilton Circuit

The original puzzle, attributed to Sir William Rowan Hamilton, consisted in finding a travel route along the edges of a regular dodecahedron, where each vertex represented a different city, in such a way as to pass through each city

just once. Instead of the dodecahedron one can use a planar graph isomorphic to the graph formed by the edges of the dodecahedron. Thus a Hamilton line in a graph is a circuit that passes through each vertex exactly once, although it does not, in general, cover all the edges.

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4.5 Lattice Problems; Taxicab Geometry

In ancient cities, streets often followed the contours of hills or the meanderings of cowpaths. In modern times, some deliberate pattern is usually discernible: thus in Washington, D. C., streets run radially from a central hub; in Philadelphia, the pattern is based on generous-sized squares.

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4.6 Map-Coloring Problems

"Considering the fame and tender age of the four-color conjecture, our knowledge of its origins is surprisingly vague. . . . If cartographers are aware of the four-color conjecture, they have certainly kept the secret well. But their lack of interest is quite understandable. . . . Moreover, the coloring of a geographical map is quite different from the formal problem posed by mathematicians. . . . The four-color conjecture cannot claim either origin or

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4.8 The Moebius Strip

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And you'll get quite a laugh
If you cut one in half
For it stays in one piece when divided.

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4.9 Paper Folding—Origami

Folding paper to form plane geometric figures and three-dimensional objects implies that only paper and pencil are used—no scissors or other tools. Thus geometric paper folding differs from toy paper folding. Thus it is possible, by simply folding and creasing, to perform all the basic constructions of plane Euclidean geometry that can be executed with the compass and straightedge.

Origami, the age-old Japanese art of paper folding, is concerned with creating picturesque objects, say a lantern, a swan, a fan, or a sailboat. Some Japanese children become so expert that they can make an object by starting with a piece of paper one inch square. In recent years, origami has aroused considerable interest in America.

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4.10 Polyominoes; the Soma Cube

Polyominoes were unknown until 1954, when they were introduced by Solomon Golomb in an address before the Mathematics Club at Harvard University. Since then they have become widely known, appealing to puzzle enthusiasts and professional mathematicians alike.

The *Soma* cube was invented by the contemporary Danish writer Piet Hein, who also invented the games of *Hex* and *Tac Tix.* The *Soma* pieces more or less comprise a three-dimensional analog of the Chinese tangram. *Soma* constructions involve spatial imagination and raise many interesting problems in combinatorial geometry.

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4.11 Polytopes

Strictly speaking, polytopes and polyhedra are geometrical rather than topological figures, and have been included here only for convenience. A *polytope* is a geometrical figure bounded by portions of lines, planes, or

hyperplanes; in two dimensions, it is a polygon; in three dimensions, a polyhedron. Beside tessellations and star polygons, regular, semi-regular, and star polyhedra, generalized polytopes include zonohedra, kaleidoscopes, and figures in spaces of more than three dimensions. Graph theory, networks, and group theory are intimately associated with the general theory of polytopes.

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4.12 Regular Polyhedrons

The five so-called Platonic solids, regarded by the Greeks as perfect figures, were destined to play a role in cosmogony. For the ancients, four of these solids represented the primordial elements: *earth, fire, water, air;* the fifth,

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eleven distinct uniform plane tessellations are possible; only three of these are regular, i.e., all the polygons involved (tiles) are regular polygons, and identical. In the remaining eight tessellations, which involve the use of unlike regular tiles, the same number and kind of polygons (triangles, squares, etc.) are used at each vertex. For example, the tessellation with four triangles and a hexagon at each vertex has some interesting properties.

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Chapter 4.

4.14 Topological Recreations in General

In 1872 Felix Klein classified geometries according to the groups of transformations under which their theorems remain true. In this sense, projective geometry is more general than affine geometry, and topology is more general than projective geometry. Topology is characterized by the group of continuous transformations, i.e., transformations that preserve what are technically known as neighborhoods. Such transformations may be visualized as permitting any amount of stretching or compressing without tearing, which accounts for the epithet "rubber-sheet geometry."

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Chapter 5

Magic Squares

5.1 Magic Squares in General

"The peculiar interest of magic squares and all *lusus numerorum* in general lies in the fact that they possess the charm of mystery. They appear to betray some hidden intelligence which by a preconceived plan produces the impression of intentional design, a phenomenon which finds its close analogue in nature."—Paul Carus, Introduction to *Magic Squares and Cubes,* by W. S. Andrews.

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5.2 Magic Squares with Special Properties

"Magic squares form only one of many interesting questions dealing with the disposition of objects in rectangular arrays of cells. Chess and checkers come to mind at once as examples, and we shall find many other games and problems that are, or may be, based on the same idea . . . the lattice. . . . By a lattice is [here] meant a set of points at a finite distance from each other, regularly spaced throughout a space of any number of dimensions."—Maurice Kraitchik, *Mathematical Recreations.*

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5.4 The Magic Knight's Tour

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sive moves cover each square once and only once. A closely related and somewhat more formidable problem is to devise a "magic" knight's tour on any square board such that, if the successive squares visited by the knight are numbered 1, 2, 3, . . . , n^2, the resulting array becomes a magic square.

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5.5 Latin Squares and Euler Squares

Although neither of these are magic squares, it is convenient to include them at this point.

A Latin square of the n^{th} order is a square array of n^2 cells (n rows and n columns) in which n^2 letters consisting of n a's, and n b's, . . . are arranged in the cells so that the n letters in each row and in each column are different.

An Euler square of order n is a square in which the cells are filled with n elements of one kind: $a_1, a_2, \ldots a_n$ and n elements of a second kind: $b_1\ b_2, \ldots b_n$ in such a way that—

1. each cell contains one element of each kind;
2. each element of the first kind is paired with each element of the second kind exactly once;
3. each row and each column contains all the elements of both kinds.

An Euler square may thus be regarded as an appropriate combination of two Latin squares.

Euler squares are also known as Graeco-Latin squares. These squares evolved from a famous problem of Euler's, according to which thirty-six officers of six different ranks and six different regiments are to be arranged in a square so that each row and column contained six officers from different regiments and of different ranks. The problem has no solution, since it has been shown that no Euler squares of order 6 exist.

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Chapter 6

The Pythagorean Relation

6.1 Pythagorean Recreations

"The Pythagorean web of ideas does not form a precise or perfect design. There are dangling threads and ragged edges. But the whole thing hangs together, and even the loose threads may gather dew drops which light up in the sun."—Evans G. Valens, *The Number of Things; Pythagoras, Geometry and Humming Strings.*

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6.2 The Pythagorean Theorem

"No other proposition of geometry has exerted so much influence on so many branches of mathematics as has the simple quadratic formula known as the Pythagorean theorem. Indeed, much of the history of classical mathematics, and of modern mathematics, too, for that matter, could be written around that proposition."—Tobias Dantzig, *The Bequest of the Greeks.*

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6.3 Pythagorean Triples

If any two terms of a Pythagorean triple are relatively prime, then the triple is primitive; e.g., 3, 4, 5; 5, 12, 13; 8, 15, 17. Any integer which divides two terms of a Pythagorean triple also divides the third term. It is no

surprise that the properties of Pythagorean triples are closely related to number theory. For example: although the Greeks knew that the hypotenuse of a primitive triple is always an odd integer, we now know that for an odd integer R to be the hypotenuse of a primitive triple, a necessary and sufficient condition is that every prime divisor of R be of the type $4n + 1$.

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Chapter 7

Classical Problems of Antiquity

7.1 Computation of Pi (π)

"But just such relations between infinite series and π illustrate the profound connection between most mathematical forms, geometric or algebraic. It is mere coincidence, a mere accident that π is defined as the ratio of the circumference of a circle to its diameter. No matter how mathematics is approached, π forms an integral part."—Edward Kasner and James R. Newman, *Mathematics and the Imagination.*

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7.2 History and Nature of Pi (π)

"I recollect a distinguished professor explaining how different would be the ordinary life of a race of beings born, as easily they might be, so that the fundamental processes of arithmetic, algebra and geometry were different from those which seem to us so evident; but, he added, it is impossible to conceive of a universe in which ϵ and π should not exist."—W. W. Rouse Ball, *Mathematical Recreations and Essays.*

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7.3 Pi and Probability

"One day, explaining to [a friend] how it should be ascertained what the chance is of the survivors of a large number of persons now alive lying between given limits of number at the end of a certain time, I came, of course, upon the introduction of π, which I could only describe as the ratio of the circumference of a circle to its diameter. 'Oh, my dear friend! that must be a delusion; what can the circle have to do with the numbers alive at the end of a given time?' "—Augustus De Morgan, *A Budget of Paradoxes.*

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"The pseudomath is a person who handles mathematics as a monkey handles the razor. The creature tried to shave himself as he had seen his master do; but, not having any notion of the angle at which the razor was to be held, he cut his own throat. He never tried it a second time, poor animal! but the pseudomath keeps on in his work, proclaims himself clean shaved, and all the rest of the world hairy."—Augustus De Morgan, *A Budget of Paradoxes.*

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7.5 Trisecting an Angle—Duplicating the Cube

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Chapter 8

Combinatorics and Probability

8.1 Permutation and Combinatorial Problems

"Many of the problems studied in the past for their amusement or aesthetic appeal are of great value today in pure and applied science. Not long ago finite projective planes were regarded as a combinatorial curiosity. Today they are basic in the foundations of geometry and in the analysis and design of experiments. Our new technology with its vital concern with the discrete has given the recreational mathematics of the past a new seriousness of purpose."—H. J. Ryser, *Combinatorial Mathematics.*

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8.2 Tournament Problems, Round Robins

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"The mind has its illusions as the sense of sight; and in the same manner as the sense of feeling corrects the latter, reflection and calculation correct the former. Probability based upon daily experience, or exaggerated by fear and by hope, strikes us more than a superior probability, but it is only a simple result of calculus. . . . Our passions, our prejudices, and dominating opinions, by exaggerating the probabilities which are favorable to them and by attenuating the contrary probabilities, are the abundant sources of dangerous illusions."—Pierre Simon de Laplace, *A Philosophical Essay on Probabilities.*

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8.4 Games of Chance; Gambling

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Chapter 9

Manipulative Recreations

9.1 Tic-Tac-Toe

Also known as noughts and crosses, the game of tic-tac-toe, one of the oldest and simplest of amusements for two players, lends itself to neat mathematical analysis of game strategy. Several modifications of the game have evolved, such as Mill, or Nine Men's Morris, and, in recent times, three-dimensional and hyperdimensional tic-tac-toe, as well as electronic tic-tac-toe devices.

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9.2 The Fifteen Puzzle

The Fifteen Puzzle, also known as the Boss Puzzle (in French, *diablotin* or *jeu de taquin*), became popular in Europe about 1880. Its origin is uncertain, although it has been attributed to Sam Loyd (1878). Square counters numbered 1 to 15 are placed in a shallow square tray which holds just sixteen such counters. Initially placed in the tray in random order, the puzzle is to rearrange them (by sliding only) in numerical order, with the blank space remaining in the lower right hand corner. If the blank space has to be "moved" through an odd number of spaces, the solution is impossible.

Theoretically, the puzzle can be extended to a tray of $m \times n$ spaces with

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9.3 Solitaire Games

The game of Solitaire, a popular recreation in Britain, the United States, and the Soviet Union, consists of a board with thirty-three cells. Also known as Peg Solitaire, it is somewhat similar to "Chinese Checkers," in which the board is star-shaped rather than cruciform.

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9.4 Nim; Wythoff's Game

Another well known recreation for two players, Nim, takes a variety of forms. In general, any number of coins or counters are arranged arbitrarily into several piles. Players alternately remove one or more counters from any one of the piles; the player drawing the last counter (or counters) wins. Sounds simple! But the game lends itself admirably to mathematical analysis, and is closely related to the binary system of numeration.

One variation of the game consists of twelve counters arranged in three rows, with three, four, and five counters in the rows. Another variation is Wythoff's game, in which there are only two piles; in each draw the player may take counters, as he wishes, from either pile or from both piles, but in the latter instance he must draw the same number from each pile. He who takes the last counter wins.

Both mechanical and electronic devices have incorporated the game of Nim.

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Nim and Tac Tix, pp. 151–61.

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Brief discussion of a mechanical device for playing Nim; accompanying booklet explains the how and why of Nim, both informally and in terms of Boolean algebra.

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"Mathematical games," pp. 9–34. Deals primarily with the binary scale, Nim, the 12-coin problem, and related recreations.

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Game of Nim, pp. 144–54.

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Wythoff, W. A. A modification of the game of NIM. *Nieuw Archief voor Wiskunde* (Groningen) 7:199–202; 1907.

9.5 Board Games and Amusements

But helpless Pieces of the Game He plays
Upon this Chequer-board of Nights and Days;
Hither and thither moves, and checks, and slays,
And one by one back in the Closet lays.

Omar Khayyám, *The Rubáiyát* (FitzGerald trans.), st. 69

Abbott, Robert. ??? *Rec.M.M.*, no. 10, pp. 29–34; Aug. 1962.

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Describes 91 games from many lands, going back about 5,000 years.

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Brown, John R. Extendapawn—an inductive analysis. *M.Mag.* 38:286–99; Nov. 1965.

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Chessboard problems, pp. 109–19.

Engel, Douglas. Quantum chess. *Pentagon* 27:99–103; Spring, 1968.

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The Knight's Tour, pp. 309–18; 345–56.

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Gardner, Martin. *The Unexpected Hanging.* New York: Simon & Schuster, 1969.

"The Eight Queens and Other Chessboard Diversions," pp. 186–97.

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Iyer, M. R., and Menon, V. V. On coloring the $n \times n$ chessboard. *Am.M.Mo.* 73:721–25; Aug.–Sept. 1966.

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Lucas, Edouard. *Récréations Mathématiques.* Vol. 1. Paris, 1891.
Le problème des huit reines au jeu des éches, pp. 59–86.

Madachy, Joseph. *Mathematics on Vacation.* New York: Charles Scribner's Sons, 1966.
"Chessboard placement problems," pp. 34–54.

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A game for two, played on a matrix of 25 squares.

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Scarne, John. *Scarne on Teeko.* New York: Crown Publishers, 1955. 256 pp.

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Yaglom, A. M., and Yaglom, I. M. *Challenging Mathematical Problems with Elementary Solutions.* San Francisco: Holden-Day, 1964.

Section III: Chessboard problems.

Yarbrough, L. D. Uncrossed knight's tours. *J.R.M.* 3:140–42; July 1968.

9.6 Manipulative Games and Puzzles

In this category are included miscellaneous recreations involving the manipulation of objects, for example, coins, matchsticks, the Chinese rings, the Tower of Hanoi, dominoes, Oware and similar pebble games, the game of Odds, and so on. The classification is admittedly loose, for the Fifteen puzzle, Nim, Solitaire, and pastimes relating to objects changing places would also fall into this category.

Abraham, R. M. *Easy-to-do Entertainments and Diversions,* etc. New York: Dover Publications, 1961. (Paper)

Coin tricks and games, pp. 49–56; Tricks with matches, pp. 57–68.

Albaugh, A. H. Game of Euclid. *M.T.* 54:436–39; Oct. 1961.

A card game, similar to gin rummy, in which the steps in a proof are matched with the "reasons" for the steps.

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Callaway, Rhona. A game in maths; how you play tower tops. *Math.Tchg.,* no. 40, pp. 42–43; Autumn 1967.

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Descriptions of many versions of the stone game of Wari as played in various countries of Asia and Africa.

Crowe, D. W. The n-dimensional cube and the Tower of Hanoi. *Am.M.Mo.* 63:29–30; Jan. 1956.

Crowther, P. A., and Hardcastle, R. A. The game of odds; a follow-up. *Math.Tchg.*, no. 47, pp. 31–33; Summer 1969.

Dohler, Dora. The role of games, puzzles, and riddles in elementary mathematics. *A.T.* 10:450–52; Nov. 1963.

Domoryad, A. P. *Mathematical Games and Pastimes.* New York: Pergamon Press, 1964.

Manipulative games and games with piles of objects, pp. 61–78; Pastimes with dominoes, pp. 107–8.

Field, Paul B. A description of a math field day. *S.S.M.* 64:12–14; Jan. 1964.

Includes brief discussion of Hex, Nim, Five-in-a-Row, and Three-Dimensional Tic-Tac-Toe.

Gardner, Martin. *New Mathematical Diversions from Scientific American.* New York: Simon & Schuster, 1966.

Bridg-it and Other Games, pp. 210–18.

Gardner, Martin. *The Second Scientific American Book of Mathematical Puzzles and Diversions.* New York: Simon & Schuster, 1961.

On Bridg-it, pp. 84–87.

Gardner, Martin. Mathematical games. *Sci.Am.* 206:140–43; Mar. 1962.

The game of hexapawn.

Gardner, Martin. Assorted puzzles and tricks. *Sci.Am.* 219:106–11; Aug. 1968.

Modified ticktacktoe; 16-penny puzzles; cube dissection; topological puzzle; etc.

Gardner, Martin. Game theory applied to games. *Sci.Am.* 217:127–32; Dec. 1967.

Gardner, Martin. Magical tricks based on mathematical principles. *Sci.Am.* 211:96–99; Aug. 1964.

Three-cup trick; "think-a-word" trick; Yates' 12-penny trick; Bowman's dollar-bill trick.

Gardner, Martin. Mathematical strategies for two-person contests. *Sci.Am.* 216:116–20; Feb. 1967.

"Isomorphism" of such games: e.g., game of Jam, Hot, and modified ticktacktoe.

Gardner, Martin. Penny puzzles: a collection of coin problems. *Sci.Am.* 214:112–18; Feb. 1966.

Grundy, P. M. Mathematics and games. *Eureka* 2:6; 1939. Reprint, 27:9; 1964.

Hadley, Judy. Functions occurring in puzzle and games. *Math.Tchg.*, no. 42, pp. 45–51; Spring 1968.

Haggerty, J. B. Kalah: an ancient game of mathematical skill. *A.T.* 11:326–30; May 1964.

Well-known game involving all four fundamental processes as well as pure reasoning; also known as Oware.

Hamilton, Joseph M. C. The game of pebbles. *Los Angeles Mathematics Newsletter*, vol. 3, no. 2, p. 5; Jan. 1956.

Harris, P. A. Mathematical bingo. *M.T.* 54:577–78; Nov. 1961.

Similar to conventional bingo, in which the solution sets of given equations must be identified.

Hemmings, Ray. Attribute materials. *Math.Tchg.*, no. 37, pp. 10–16; Winter 1966.

Games that can be played by manipulating blocks of various shapes and colors, and which may lead to discovery of mathematical patterns and relations.

Lucas, Edouard. *Récréations Mathématiques*, Paris, 1891.

Le jeu de dominos, vol. 2, pp. 39–71; La géométrie des réseaux et le problème des dominos, vol. 4, pp. 123–51.

Lucas, Edouard. *Récréations Mathématiques*, Paris, 1891.

Les jeux de marelle, vol. 2, pp. 75–99; vol. 4, pp. 69–85.
Mill, or nine-men-morris, and variations.

Madachy, Joseph. *Mathematics on Vacation*. New York: Charles Scribner's Sons, 1966.

"Dominoe recreations," pp. 209–19.

McIntosh, Alistair. A puzzle untangled. *Math.Tchg.*, no. 46, pp. 16–18; Spring 1969.

Mosteller, F. Optimal length of play for a binomial game. *M.T.* 54:411–12; Oct. 1961.

Mygaard, P. H. Odd and even; a game. *M.T.* 49:397; May 1956.

Ogilvy, C. Stanley. *Tomorrow's Math*. Oxford University Press, 1962. 182 pp.

Problems concerning games, pp. 37–50.

Philpott, Wade. Quadrilles. *Rec.M.M.*, no. 14, pp. 5–11; Jan.–Feb. 1964.

Domino patterns.

Pits and Pebbles. *Time*, June 14, 1963; p. 67.

Punga, M. Le jeu Ruma. *Sphinx*, 1931; pp. 113–15.

Sawyer, W. W. Analysis of an Indian game. *Scrip.M.* 22:71–78; 1956.

Game similar to Fox and Geese.

Sawyer, W. W. The game of Oware. *Mathematics News Letter*, Sept. 1958; *Scrip.M.*, vol. 15, no. 2, pp. 159–61; June 1949.

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Domino puzzles, pp. 38–68.

Smith, Cedric A. B. Compound games with counters. *J.R.M.* 1:67–77; Apr. 1968.

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A game also known as the "Pebble Game" (Dudeney, *Amusements in Mathematics*).

Wells, Celia, and Wells, Peter. A Sunday afternoon with five cocktail sticks. *Math.Tchg.*, no. 47, pp. 52–53; Summer 1969.

Williams, Russell. Bingtac. *A.T.* 16:310–11; Apr. 1969.

A game for junior high school level, similar to the game of "Yahoo."

9.7 Card Games and Card Tricks

An ordinary deck of 52 playing cards lends itself to a fantastic variety of tricks—tricks based on their numerical values, tricks based on different colors and different suits, tricks based on the cards as counting units, shuffling tricks, locating and/or naming the position of a card, and so on and on. According to Martin Gardner, although "cards were used for gaming purposes in ancient Egypt, it was not until the fourteenth century that decks could be made from linen paper, and not until the early fifteenth century that card-playing became widespread in Europe. Tricks with cards were not recorded until the seventeenth century and books dealing with card magic did not appear until the nineteenth."

Abbott, Robert. *Abbott's New Card Games.* New York: Funk & Wagnalls, 1968. 138 pp. (Paper)

Abraham, R. M. *Easy-to-do Entertainments and Diversions,* etc. New York: Dover Publications, 1961. (Paper)

Card tricks, pp. 1–19.

Abraham, R. M. *Winter Nights Entertainments.* New York: E. P. Dutton & Co., 1933.

Card tricks, pp. 1–25.

Amir-Moéz, Ali R. Limit of a function and a card trick. *M.Mag.* 38:191–96; Sept. 1965.

Amir-Moéz, Ali R. Mathematics and cards. *Rec.M.M.*, no. 8, pp. 40–42; Apr. 1962.

About card games.

"Berkeley" and T. B. Rowland. *Card Tricks and Puzzles.* 1892. 120 pp.

Chiefly mathematical tricks and puzzles; magic squares, magic tour; 15-puzzle; etc.

Card Trick Involving Pascal's Triangle. *A.T.* 15:265, 268; Mar. 1968; also, see Martin Gardner, "Mathematical Games," in *Scientific American,* Dec. 1966.

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Mathematical analysis of shuffling a deck of cards.

Gardner, Martin. *Mathematics, Magic and Mystery.* New York: Dover Publications, 1956.

Chapters 1–2: "Tricks with Cards," pp. 1–32.

Gardner, Martin. *New Mathematical Diversions from Scientific American.* New York: Simon & Schuster, 1966.
Victor Eigen: Mathemagician; pp. 103–12; Card tricks, etc.

Goldsmith, Colin. Variations on a theme. *Math.Tchg.*, no. 40, p. 48; Autumn 1967.
An extension of a well-known card trick.

Hunt, Martin. Arithmetic card games. *A.T.* 15:736–38; Dec. 1968.

Johnson, Donovan. Bridget, an algebra card game. *M.T.* 51:614–15; Dec. 1958.

Johnson, Donovan. Mathematics rummy. *M.T.* 52:373–75; May 1959.
A card game.

Kirkpatrick, P. H. Probability theory of a simple card game. *M.T.* 47:245–48; Apr. 1954.

Saunders, K. D. Shuffles. *Math.Tchg.*, no. 46, pp. 19–21; Spring 1969.
An intriguing card trick.

Simon, William. *Mathematical Magic.* New York: Charles Scribner's Sons, 1964.
Magic with playing cards. pp. 156–83.

9.8 Colored Squares and Cubes

Many versions of the "colored cubes" puzzle abound. One familiar form involves four cubes which have to be assembled in a block of $4 \times 1 \times 1$ with four different colors or designs showing on all four sides of the assembly. These puzzles have appeared in recent years on both sides of the Atlantic under such trade names as the "Tantalizer," the Mayblox puzzle, the "Katzenjammer Puzzle," "Instant Insanity," and others. A complete analysis involves considerable mathematics.

Abraham, R. M. *Diversions and Pastimes.* New York: Dover Publications, 1964.
Puzzle of the four colored cubes, p. 100.

Ball, W. W. R. *Mathematical Recreations and Essays.* Macmillan, 1960.
Colour-cube problem, pp. 112–14.

Brown, T. A. A note on "Instant Insanity." *M.Mag.* 41:167–69; Sept. 1968.
Analysis of a contemporary puzzle involving four multicolored unit cubes which are to be assembled into a $1 \times 1 \times 4$ rectangular prism so that all four colors appear on each of the four long faces of the prism.

Carteblanche, F. de. The coloured cubes problem. *Eureka* 9:9; 1947.

Ehrenfeucht, Aniela. *The Cube Made Interesting.* New York: Pergamon Press, 1964.
"Coloured Blocks" and "Constructions from Coloured Blocks," pp. 46–66.

Farrell, Margaret. The Mayblox problem. *J.R.M.* 2:51–56; Jan. 1969.
A set of eight colored cubes; puzzle invented by Major P. A. MacMahon.

Filipiak, Anthony. *100 Puzzles.* New York: A. S. Barnes, 1942.
Colored-cubes puzzle, p. 108.

Gardner, Martin. *New Mathematical Diversions from Scientific American.* New York: Simon & Schuster, 1966.
The 24 Color Squares and the 30 Color Cubes: pp. 184–95.

Gardner, Martin. McMahon's 24-color triangles; cube problems; etc. *Sci.Am.* 219:120–25; Oct. 1968.

Johnson, Paul B. Stacking colored cubes. *Am.M.Mo.* 63(6):392–95; June–July 1956.

Kraitchik, Maurice. *Mathematical Recreations,* New York: Dover Publications, 1953.
Colored squares, pp. 312–13.

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Perisho, Clarence. Colored polyhedra: a permutation problem. *M.T.* 53:253–55; Apr. 1960.

Philpott, Wade. MacMahon's three-color squares. *J.R.M.* 2:67–78; Apr. 1969.

Winter, Ferdinand. *Das Spiel der 30 Bunten Würfel.* Leipzig. 1934. 128 pp.
Devoted exclusively to the 30 colored cubes problem.

9.9 Mechanical Puzzles

In earlier times, puzzles involving principles of elementary physics and mechanics were frequently included with mathematical puzzles and recreations. Thus for example, the famed *Mathematical Recreations* of Jean Leurechon (1633), as attested in the title, dealt among other topics with "opticks, statick, mechanicks, chymistry, water-works, fire-works." Similarly, William Leybourn's *Pleasure With Profit: Recreations of Divers Kinds* (1694), reveals on the title page "Numerical, Geometrical, Mechanical, Statical, Astronomical, Horometrical, Cryptographical, Magnetical, Automatical, Chymical, and Historical."

Filipiak, Anthony. *101 Puzzles: How to Make and How to Solve Them.* New York: A. S. Barnes & Co., 1942.

Gardner, Martin. *The Second Scientific American Book of Mathematical Puzzles and Diversions.* New York: Simon & Schuster, 1961.
"Mechanical Puzzles." pp. 210–19.

Gardner, Martin. Puzzles that can be solved by reasoning based upon principles of physics. *Sci.Am.* 215:96–99; Aug. 1966.

Gardner, Martin. A puzzling collection. *Hobbies,* Sept. 1934, p. 8.

Hemmings, Ray. Think-a-dot. *Math.Tchg.*, no. 40, p. 45; Autumn 1967.

Brief description of an ingenious device, which, while intriguing, can lead to serious and significant generalizations.

Kravitz, Sidney. Additional mathematical theory of "Think-a-Dot." *J.R.M.* 1:247–50; Oct. 1968.

Lehman, Alfred. A solution of the Shannon switching game. *Journal, Soc. Industrial Applied Math.*, vol. 12, no. 4, pp. 687–725; Dec. 1964.

Materials for Mathematics: Games and Puzzles. *Math.Tchg.*, no. 40, p. 32; Autumn 1967.

Gives a list of about 35 general recreational devices commercially available to illustrate such matters as tessellations; permutation puzzles; interlocking solid dissections; networks; polyominoes; etc.

Professor Hoffmann (pseudonym of Angelo Lewis). *Puzzles Old and New.* New York: Frederick Warne & Co., 1893.

Schuh, Fred. *The Master Book of Mathematical Recreations.* New York: Dover Publications, 1968.

Puzzles in Mechanics, pp. 390–430; Puzzles in dynamics; kinematics; inertia; forces; mass and weight.

Schwartz, Benjamin L. Mathematical theory of think-a-dot. *M.Mag.* 40:187–93; Sept. 1967.

Slocum, Jerry. Making and solving puzzles. *Science and Mechanics*, Oct. 1955; pp. 121–26.

Stubbs, A. Duncan. *Miscellaneous Puzzles.* New York: Frederick Warne & Co., 1931.

Interesting and unusual mechanical puzzles.

9.10 Mathematical Models

The making of mathematical models is a fascinating challenge to many people. It need not be limited to models of regular and semiregular solids, although these are rather popular. It can be carried on at various levels, depending upon the maturity of the model-maker as well as his skill and artistry.

Bruyr, Donald. *Geometrical Models and Demonstrations.* Portland, Me.: J. Weston Walch, 1964. 173 pp.

Curves, surfaces, solids, instruments, etc.; over 150 diagrams.

Cameron, A. J. *Mathematical Enterprises for Schools.* New York: Pergamon Press, 1966. 187 pp.

Nets and Solids, pp. 33–45.

Cundy, H. Martyn, and Rollett, A. P. *Mathematical Models.* Oxford University Press, 1967. 286 pp.

Hess, Adrien L. *Mathematics Projects Handbook.* Boston: D. C. Heath & Co., 1962. 60 pp. (Paper)

Kenna, L. A. *Understanding Mathematics with Visual Aids.* Totowa, N.J.: Littlefield, Adams & Co., 1962. 174 pp. (Paper)

Discusses wooden models, string models, curve-stitching, paper-folding, etc.

Mathematics in Kensington. *Math.Tchg.*, no. 23, pp. 9–13; Summer 1963.

Describes models of surfaces formed by complex algebraic equations, to be seen in the Science Museum in Kensington.

Meredith, G. Patrick. *Algebra by Visual Aids.* 4 vols. Allen, 1948.

Bk. 1: The Polynomials; Bk. 2: The Continuum; Bk. 3: The Laws of Calculation; Bk. 4: Choice and Chance.

Chapter 10

Miscellaneous Recreations

10.1 Logical Paradoxes

"So far as mathematics as a whole is concerned, the setbacks occasioned by the paradoxes of logic have been more than balanced by the advances resulting from their subsequent investigation."—Eugene Northrop, *Riddles in Mathematics.*

Alexander, Peter. Pragmatic paradoxes. *Mind* 59:536–38; Oct. 1950.

Bernhard, Robert. Crisis in math—is there a "universal truth?" *Scientific Research* 3:47–56; Oct. 14, 1968.

Mathematical paradoxes.

Chapman, J. M., and Butler, R. J. On Quine's "so-called paradox." *Mind* 74:424–25; July 1965.

Cohen, L. Jonathan. Mr. O'Connor's "Pragmatic Paradoxes." *Mind* 59:85–87; Jan. 1950.

Ebersole, F. B. The definition of "pragmatic paradox." *Mind* 62:80–85; Jan. 1953.

Gardner, Martin. A new paradox, and variations on it, about a man condemned to be hanged. *Sci.Am.*, Mar. 1963, p. 144.

Gardner, Martin. *The Unexpected Hanging.* New York: Simon and Schuster, 1969.

"The Paradox of the Unexpected Hanging," pp. 11–23.

Kiefer, James, and Ellison, James. The prediction paradox again. *Mind* 74:426–27; July 1965.

Lyon, Ardon. The prediction paradox. *Mind* 68:510–17; Oct. 1959.

Maxfield, Margaret. How quaint the ways of paradox. In *Chips from the Mathematical Log.* Mu Alpha Theta, 1966; pp. 59–60.

Medlin, Brian. The unexpected examination. *American Philosophical Quarterly* 1:1–7; Jan. 1964.

Meltzer, B. The third possibility. *Mind* 73:430–33; July 1964.

Meltzer, B., and Good, I. J. Two forms of the prediction paradox. *British Journal for the Philosophy of Science* 16:50–51; May 1965.

Nerlich, G. C. Unexpected examinations and unprovable statements. *Mind* 70:503–13; Oct. 1961.

O'Beirne, T. H. Can the unexpected *never* happen? *New Scientist* 15:464–65; May 25, 1961; pp. 597–98; June 8, 1961.

O'Connor, D. J. Pragmatic paradoxes. *Mind* 57:358–59; July 1948.

O'Connor, D. J. Pragmatic paradoxes and fugitive propositions. *Mind* 60:536–38; Oct. 1951.

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Schoenberg, Judith. A note on the logical fallacy in the paradox of the unexpected examination. *Mind* 75:125–27; Jan. 1966.

Scriven, Michael. Paradoxical announcements. *Mind* 60:403–7; July 1951.

Sharpe, R. A. The unexpected examination. *Mind* 74:255; Apr. 1965.

Shaw, R. The paradox of the unexpected examination. *Mind* 67:382–84; July 1958.

Van Heerden, P. J. Logical paradoxes are acceptable in Boolean algebra. *M.Mag.* 39:175–78; May 1966.

Weiss, Paul. The prediction paradox. *Mind* 61:265–69; Apr. 1952.

Woodall, D. R. The paradox of the surprise examination. *Eureka*, no. 30, pp. 31–32; Oct. 1967.

Wright, J. A. The surprise exam: prediction on last day uncertain. *Mind* 76:115–17; Jan. 1967.

10.2 Logical and Inferential Problems

Many people find these puzzles the most tantalizing of all—whether they be like Lewis Carroll's fantastic syllogisms, or conundrums of the Smith-Jones-Robinson type, or the colored-hat type question, or the brand of truth-telling and lying puzzles. Not infrequently the solution of such recreational logic problems is facilitated by the use of matrices, truth tables, or set theory.

Adler, Irving. *Logic for Beginners through Games, Jokes and Puzzles.* New York: John Day Co., 1964. 158 pp.

Rather elementary and prosaic.

Allen, L. E. Toward autotelic learning of mathematics. *M.T.* 56 8–21; Jan. 1963.

Games involving logical inference; a report on Wff'n Proof games.

Brown, David H. The problem of the three prisoners. *M.T.* 59:131–32; Feb. 1966.

Buchalter, Barbara. The logic of nonsense. *M.T.* 55:330–33; May 1962.

Delightful discussion of the logic in *Alice in Wonderland.*

Carroll, Lewis (C. L. Dodgson). *Mathematical Recreations of Lewis Carroll.* Vol. 1: Symbolic Logic, and The Game of Logic. (2 books bound as 1.) New York: Dover Publications, 1958. 199 + 69 pp.

The first book consists of some 400 logical problems involving syllogisms and sorites.

Carroll, Lewis (C. L. Dodgson). *Mathematical Recreations of Lewis Carroll.* Vol. 2: Pillow Problems, and A Tangled Tale. (2 books bound as 1.) New York: Dover Publications, 1958. 109 + 152 pp.

"Pillow Problems" is a classical collection of 72 sophisticated "brain-teasers."

Dienes, Z. P., and Golding, E. W. *Learning Logic, Logical Games.* New York: Herder & Herder, 1966. 80 pp. (Paper)

Domoryad, A. P. *Mathematical Games and Pastimes.* New York: Pergamon Press, 1964.

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Gardner, Martin. *Logic Machines and Diagrams.* New York: McGraw-Hill Book Co., 1958. 157 pp. (Paper)

Gardner, Martin. *The Second Scientific American Book of Mathematical Puzzles and Diversions.* New York: Simon & Schuster, 1961.

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Hunter, J. A. H. Some inferential problems. *Rec.M.M.*, no. 1, pp. 3–6; Feb. 1961.

Solving problems in logic with Boolean algebra.

Pedoe, Dan. *The Gentle Art of Mathematics.* English Universities Press, 1958.

"Automatic thinking," pp. 65–76; "Double Talk," pp. 128–35. Syllogisms, logical and inferential problems; logical paradoxes.

Phillips, Hubert [Caliban]. *My Best Puzzles in Logic and Reasoning.* New York: Dover Publications, 1961. 107 pp.

An excellent collection of "logic problems," almost all original.

Summers, George. *Fifty Problems in Deduction.* New York: Dover Publications, 1969.

Summers, George. *New Puzzles in Logical Deduction.* New York: Dover Publications, 1968. 121 pp. (Paper)

Williams, Horace. Constructing logic puzzles. *M.T.* 54:524–26; Nov. 1961.

Wylie, Clarence R. *101 Puzzles in Thought and Logic.* New York: Dover Publications, 1957. Unpaged. (Paper)

10.3 Cryptography—Cryptanalysis—Codes and Ciphers

"In point of fact, as I have found by experience, in cryptography the exceptions are more frequent than the rule."—André Langie, *Cryptography.*

Andree, Richard. Cryptography. In *Chips from the Mathematical Log.* Mu Alpha Theta, 1966; pp. 25–26.

Friedman, W. F., and Mendelsohn, C. J. A bibliography of cryptography. *Am.M.Mo.* 50:345; 1943.

Gaines, Helen F. *Cryptanalysis. A Study of Ciphers and Their Solution.* New York: Dover Publications, 1956. 237 pp.

Unabridged and corrected edition of a work formerly published in *Elementary Cryptanalysis.* Solutions of about 150 specimen codes are included.

Gaines, Helen F. *Elementary Cryptanalysis.* Boston: American Photographic Publishing Co., 1943.

Glaymann, Maurice. Un modèle d'espace vectoriel et son utilisation pour coder et decoder un message. *Journal of Structural Learning,* Jan. 1968; pp. 9–15.

Graham, L. A. *The Surprise Attack in Mathematical Problems.* New York: Dover Publications, 1968.

"Codes and Computers," pp. 28–32.

Hill, L. S. Concerning certain linear transformation apparatus in cryptography. *Am.M.Mo.* 38:135; 1921.

Hill, L. S. Cryptography in an algebraic alphabet. *Am.M.Mo.* 26:409; 1919.

Kahn, David. *The Codebreakers: The Story of Secret Writing.* New York: Macmillan Co., 1967. 1164 pp.

Nonmathematical survey of the history of secret communication, wartime cryptanalyses, etc. Exhaustive and well documented.

Kahn, David. Modern cryptography. *Sci.Am.* 215:38–46; July 1966.
Use of electronics and mathematics by cryptanalysts.

Laffin, John. *Codes and Ciphers: Secret Writings Through the Ages.* New York: Abelard-Schuman, 1964. 144 pp.

An unusually fascinating presentation, including codes from ancient to modern times.

Mann, Barbara. Cryptography with matrices. *Pentagon* 21:3–11; Fall 1961.

Mitchell, U. G. Codes and ciphers. *Am.M.Mo.* 26:409–13.

Pallas, Norvin. A little matter of espionage. *Rec.M.M.*, no. 6, pp. 56–58; Dec. 1961.

Pallas, Norvin. The Sun Dial: a cryptographic mystery. *Rec.M.M.*, no. 2, pp. 34–36; Apr. 1961.

Peck, Lyman. *Secret Codes, Remainder Arithmetic, and Matrices.* Washington, D.C.: NCTM, 1961. 54 pp. (Pamphlet)

Pratt, Fletcher. *Secret and Urgent; the Story of Codes and Ciphers.* New York: Blue Ribbon Books, 1942. 282 pp.

Shannon, C. E. Communication theory of secrecy systems. *Bell System Tech. Journal* 28:656–715; 1949.

Smith, Lawrence D. *Cryptography.* New York: Dover Publications, 1955. 164 pp.

Willerding, Margaret. Codes for boys and girls. *A.T.* 2:23–24; Feb. 1955.

Williams, Eugenia. *An Invitation to Cryptograms.* New York: Simon & Schuster, 1959.

Collection of 150 puzzles, with solutions, ranging from very easy to moderately difficult cryptograms.

Winick, David F. "Arithmecode" puzzle. *A.T.* 15:178–79; Feb. 1968.

10.4 Humor and Mathematics

Mathematics—even serious mathematics—is not all grim. Unexpected twists appeal to the humorist, as attested to by numerous limericks and other verse in a lighter vein, to say nothing of scores of cartoons, many of which have been stimulated by interest in the "new mathematics" and in electronic computers. Even some mathematicians themselves have a good sense of humor.

Birdwood, Wilbur P. *Euclid's Outline of Sex.* New York: Henry Holt & Co., 1922. 68 pp. (O.P.)

A sprightly piece of satire, poking fun at devotees of Freud.

Dudley, Patricia, et al. Further techniques in the theory of big game hunting. *Am.M.Mo.* 75:896–97; Oct. 1968.

Dunham, David. *Every Man a Millionaire.* New York: Scripta Mathematica, 1937; 95 pp.

Fadiman, Clifton. *Fantasia Mathematica.* New York: Simon & Schuster, 1958. 298 pp.

A collection of humorous stories and diversions related to mathematics.

Fadiman, Clifton. *The Mathematical Magpie.* New York: Simon & Schuster, 1962. 300 pp.

A delightful collection of humor about mathematics: aphorisms, apothegms, anecdotes, poems, limericks, cartoons, essays, and curiosa.

Gardner, Martin. *The Numerology of Dr. Matrix.* New York: Simon & Schuster, 1967. 112 pp.

Gardner, Martin. Word play. *Sci.Am.* 211:218–24; Sept. 1964.

Puns and palindromes; assorted humor.

Jablonower, Joseph. The Jabberwocky was a Lark, or the "Mathematician Takes a Holiday." *M.T.* 60:871–73; Jan. 1967.

Reprint of a brief satirical note built around Lewis Carroll characters, Alice and Humpty Dumpty.

Janicki, G. Number cartoons. *M.T.* 48:372; May 1955.

Jolly, R. F. Excelsior¿ *M.T.* 62:94–95; Feb. 1969.

A humorous poem, a parody on "Now I Am the Ruler of the Queen's Navee," from *H. M. S. Pinafore,* by Gilbert & Sullivan.

Kaufman, Gerald L. *Geo-metric Verse: Poetry Forms in Mathematics written mostly for Fanatics.* New York: The Beechhurst Press, 1948. 64 pp.

Leacock, Stephen. Human interest put into mathematics. *M.T.* 22:302–4; 1929; also, *M.T.* 59:561–63; Oct. 1966.

A humorous, whimsical spoof on stereotyped problems concerning "*A, B* and *C,* who do a certain piece of work," etc.

Leacock, Stephen. Through a glass darkly: human thought in mathematical symbols. *Atlantic Monthly* 158:94–98; 1936.

Lindon, J. A. A clerihew ABC of mathematics. *Rec.M.M.,* no. 14, pp. 24–26; Jan.–Feb. 1964.

A humorous poem.

Lindon, J. A. Numbo-carrean. *Rec.M.M.,* no. 11, pp. 11–13; Oct. 1962.

A whimsical sketch on an artificial language based on numerals and letters.

Lindon, J. A. A world of difference. *Rec.M.M.,* no. 13, pp. 31–33; Feb. 1963.

A time-space spoof.

Luke, Dorman. "Yoicks!" "Tallyho!" Shades of King Arthur! Sir Moebius rides again! *Rec.M.M.,* no. 12, pp. 13–15; Dec. 1962.

Humorous skit on the Moebius strip.

May, Kenneth O., and Anderson, P. An interesting isomorphism. *Am.M.Mo.,* vol. 70, no. 3, Mar. 1963.

Humorous, satirical sketch.

McClellan, John. Recreations for space travel. *Rec.M.M.,* no. 7, pp. 7–11; Feb. 1962.

Humor and spoofing, but suggestive.

Menninger, Karl. *Ali Baba und die 39 Kamele.* Göttingen: Vandenhoeck & Ruprecht, 1955, 1958. 108 pp. (Paper)

Humorous sketches about numbers, by the distinguished author of "Zahlwort und Ziffer."

Miller, James E. How Newton discovered the law of gravitation. *Am.M.Mo.* 69:623–31; Sept. 1962.

Clever spoof on contemporary foibles of academic administration and lavish government grants for research.

Morphy, Otto. Some modern mathematical methods in the theory of lion hunting. *Am.M.Mo.* 75:185–87; Feb. 1968.

A natural sequel to H. Petard's classic treatise on the mathematical theory of big game hunting.

Petard, H. A contribution to the mathematical theory of big game hunting. *Rec.M.M.*, no. 5, pp. 14–17; Oct. 1961. Reprinted from *Am.M.Mo.*, Aug.–Sept. 1938, pp. 446–47.

Sophisticated humor.

Sutcliffe, Alan. A walk in the rain. *Rec.M.M.*, no. 7, pp. 20–22; Feb. 1962.

Interesting curiosity on how to keep relatively dry.

Traub, Hugo. Geometry and men. *Rec.M.M.*, no. 11, pp. 6–7; Oct. 1962.

Weaver, Warren. Lewis Carroll: Mathematician. *Sci.Am.*, vol. 194, no. 4, pp. 116–28; Apr. 1956.

Willerding, Margaret F. Mathematics through a looking glass. *Scrip.M.* 25:209–19; 1960.

Delightful account of mathematical allusions in *Alice in Wonderland*, etc.; bibliography.

Winthrop, Henry. A devil's dictionary for higher education. *Rec.M.M.*, no. 9, pp. 12–15; June 1962.

Humorous skit on definitions of mathematical terms.

10.5 Sports and Mathematics

Mathematics enters the arena of sports in several unrelated ways—all of them interesting. The most obvious perhaps is the role of physics; then there are the matters of scoring and comparative records, the statistical and predictive aspects, involving probability, and the matter of tournaments. Who says one can't find mathematics almost anywhere you look for it?

A Billiard Ball Problem. [Problem E1704.] *Am.M.Mo.* 72:669; 1965.

Ap Simon, H. The luck of the toss in squash rackets. *M.Gaz.* 35:193–94; 1951.

Austin, A. K. Tennis. *Math.Tchg.*, no. 44, p. 25; Autumn 1968.

Problems about the method of scoring, the order of serving, and the changing of ends in a tennis match.

Bergman, Ronald. Something new behind the 8-ball. *Rec.M.M.*, no. 14, pp. 17–19; Jan.–Feb. 1964.

The mathematics of an elliptical pool table.

Clifford, Edward. An application of the law of sines: How far must you lead a bird to shoot it on the wing? *M.T.* 54:346–50; May 1961.

Cook, Earnshaw. *Percentage Baseball.* Cambridge, Mass.: M.I.T. Press, 1966. 416 pp.

An unusual statistical and probabilistic study of batting and pitching averages, and some implications for possible strategies.

Furucan, H. M. The velocity of sound and the start of athletic events. *Australian Mathematics Teacher* 4:4–9; 1948.

Hardy, G. H. A mathematical theorem about golf. *M.Gaz.* 29:225 ff. (Math. Note No. 1844); 1945.

Hayward, Roger. The bounding billiard ball. *Rec.M.M.*, no. 9, pp. 16–18; June 1962.

Himmelfarb, A., and Silverman, D. L. A billiard table problem. *Am.M.Mo.* 74:870; Aug.–Sept. 1967.

A billiard ball is cued from a corner of an $n \times m$-foot billiard table at an angle of 45°. How many cushions will the ball strike before it again goes into a corner?

Hemming, G. W. *Billiards Mathematically Treated.* London: Macmillan & Co., 1899. 45 pp.

Kidd, Kenneth. Measuring the speed of a baseball. *S.S.M.* 66:360–64; Apr. 1966.

Knuth, Donald. Billiard balls in an equilateral triangle. *Rec.M.M.*, no. 14, pp. 20–23; Jan.–Feb. 1964.

Lampe, Ernst. *Mathematik und Sport.* 2d ed. Leipzig: B. G. Teubner, 1956. 94 pp.

Discussion of the mathematics and physics related to throwing a ball, jumping, running, swimming, cycling, weight lifting, tennis, etc.

Madachy, Joseph. *Mathematics on Vacation.* New York: Charles Scribner's Sons, 1966.

"Bouncing billiard balls," pp. 231–41.

Maunsell, F. G. Why does a bicycle keep upright? *M.Gaz.* 30:195–99; Oct. 1946.

Miller, Fred A. In how many ways can the World Series in baseball be played? *S.S.M.* 69:71; Jan. 1969.

Ranucci, Ernest R. Anyone for tennis? *S.S.M.* 67:761–65; Dec. 1967.

Rickey, Branch. Goodbye to some old baseball ideas. *Life* 37:78–86+; Aug. 2, 1954.

Discussion of batting and pitching averages.

Rising, Gerald R. Geometric approach to field-goal kicking. *M.T.* 47:463–66; Nov. 1954.

Sarjeant, H. B. Golf from a mathematical angle. *Australian Mathematics Teacher* 3:3–9; 1947.

Walker, G. T. The physics of sport. *M.Gaz.* 20:172–77; July 1936.

10.6 Philately and Mathematics

Nearly every stamp collector knows that there are scores of topics for "thematic" collections: railroads, airplanes, bridges, ships, birds, flowers, medicine, scientists, space, etc. Among such topics, one that has not been publicized is mathematicians and mathematics on stamps.

It is of interest to note that portraits of more than one hundred different mathematicians have been depicted on postage stamps of one country or another. In addition, many physicists and astronomers have also been honored in this way. And, if one wishes, such a collection could also include stamps displaying mathematical instruments, astronomical observatories, geometric designs, mathematics in Nature, parabolas and catenaries in the form of suspension bridges and telegraph cables, ellipses and semicircles in arches, and other sundry mathematical items. An impertinent question remains to be asked: Why have so many eminent mathematicians been overlooked in the search for postal commemoratives? In particular, in the United States, why have Charles Sanders Peirce and Josiah Willard Gibbs not been honored?

Bierman, Kurt R. Comments on mathematics and philately. *M.Mag.* 34:297; 1961.

Boyer, Carl B. Philately and mathematics. *Scrip.M.* 15:105–14; 1949.

Brooke, Maxey. Mathematics and philately. *M.Mag.* 34:31–32; 1961.

"The Hamilton postage stamp." *Scrip.M.* 10:213–14; 1944.

Horton, C. W. Scientists on postage stamps. *S.S.M.* 48:445–48; 1948.

Johnson, R. A., and Archibald, R. C. Postage-stamp or coin portraits of mathematicians. *Scrip.M.* 1:183–84; 1932.

Larsen, H. D. Mathematics and philately. *Am.M.Mo.* 60:141–43; 1953.

Larsen, H. D. Mathematics on stamps. *M.T.* 48:477–80; 1955.

Larsen, H. D. Mathematics on stamps. *M.T.* 49:395–96; 1956.

Pinzka, C. F. A note on mathematics and philately. *M.Mag.* 34:169; 1961.

Schaaf, W. L. Mathematicians and mathematics on postage stamps. *J.R.M.* 1:195–216; Oct. 1968.

Check list of stamps depicting portraits of nearly a hundred mathematicians; also, stamps showing symbols, graphs, geometric designs, measurement, computation, curves, etc.; about 600 stamps in all.

Schaaf, W. L. Philately and mathematics—a further note. *M.T.* 49:289–90; 1956.

Schaaf, W. L., and Papa, John S. Scientists discovered dynamic symmetry for fine arts (etc.). *Linn's Weekly Stamp News*, June 23, 1969, pp. 8–9.

Discussion of the original paintings from which portraits of mathematicians were designed on stamps.

10.7 Assorted Recreations and Curiosities

"We may learn the same lesson . . . from the puzzle columns of the popular newspapers. Nearly all their immense popularity is a tribute to the drawing power of rudimentary mathematics, and the better makers of puzzles, such as Dudeney or 'Caliban,' use very little else. They know their business; what the public wants is a little intellectual 'kick,' and nothing else has quite the kick of mathematics."—G. H. Hardy, *A Mathematician's Apology*.

Bain, George G. The prince of puzzle-makers: An interview with Sam Loyd. *Strand Magazine* (London) 34:771–77; 1907.

Boys, C. V. *Soap Bubbles*. New York: Dover Publications, 1959.

Brider, J. E. A mathematical adventure. *Math.Tchg.*, no. 37, pp. 17–21; Winter 1966.

Finite differences when counting overlapping squares or triangles.

Brumfiel, C. F. Numbers and games. *NCTM Twenty-seventh Yearbook*, 1963; pp. 245–60.

For junior high school level: repeating decimals, continued fractions, irrational numbers, number line games, tick-tack-toe.

Clapham, J. Charles. Playful mice. *Rec.M.M.*, no. 10, pp. 6–7; Aug. 1962.

Pursuit problem.

Fletcher, T. J. Carry on Kariera! *Math.Tchg.*, no. 43, pp. 35–36; Summer 1968.

Mathematical analysis of marriage and kinship patterns.

Gardner, Martin. Nine titillating puzzles. *Sci.Am.* 197:140 ff.; Nov. 1957.

Gardner, Martin. Mathematical games—On the relation between mathematics and the ordered pairs of op art. *Sci.Am.* 213:100–105; July 1965.

Gardner, Martin. "Cooked" puzzles. *Sci.Am.* 214:122–27; May 1966.

Miscellaneous puzzles with flaws or incorrect answers, or puzzles with impossible solutions.

Gardner, Martin. Dollar bills. *Sci.Am.* 218:118–20; Apr. 1968.

Puzzles and tricks with dollar bills.

Gardner, Martin. Miscellaneous recreations. *Sci.Am.* 209:144–54; Nov. 1963;
210:114–20, June 1964;
212:112–17, Mar. 1965;
213:116–23, Nov. 1965;

216:124–29, Mar. 1967;
217:125–28, Nov. 1967;
220:124–26, Apr. 1969.

Gardner, Martin. Puzzles based on parity (odd/even). *Sci.Am.* 209:140–48; Dec. 1963.

Glenn, William, and Johnson, Donovan. *Fun with Mathematics.* New York: McGraw-Hill Book Co., Webster Div., 1960. 43 pp. (Paper)

Gogan, Daisy. A game with shapes. *A.T.* 16:283–84; Apr. 1969.
Involves rotations, symmetry, and congruence.

Grünbaum, Adolf. Are "infinity machines" paradoxical? *Science* 159:396–406; Jan. 26, 1968.

Hunter, J. A. H. The problemist at work. *Rec.M.M.*, no. 8, pp. 5–6; Apr. 1962.
General discussion.

Johnson, Donovan. Enjoy the mathematics you teach. *A.T.* 15:328–32; Apr. 1968.

Jones, L. E. Merry Christmas, Happy New Year. *S.S.M.* 67:766–71; Dec. 1967.

Langman, Harry. Curiosa: Proof of $\cos 36° - \cos 72° = \frac{1}{2}$. *Scrip.M.* 22:221; 1956.

McClellan, John. Recreations for space travel. *Rec.M.M.*, no. 7, pp. 7–11; Feb. 1962.

Perisho, C. R. Conics for Thanksgiving. *S.S.M.* 57:640–41; Nov. 1957.

Ranucci, Ernest. *Four by Four.* Boston: Houghton Mifflin Co., 1968. 60 pp.
Miscellaneous recreations involving a 4×4 network of squares.

Rapoport, A., and Rebhun, L. I. On the mathematical theory of rumor spread. *Bulletin of Mathematical Biophysics* 14:375–83; 1952.

Reeve, J. E., and Tyrell, J. A. Maestro puzzles. *M.Gaz.* 45:97–99; May 1961.
How to pack a given set of figures to form a certain figure.

Saidan, A. S. Recreational problems in a medieval arithmetic. *M.T.* 59:666–67; Nov. 1966.

Smith, Eugene P. Some puzzlers for thinkers. *NCTM Twenty-seventh Yearbook*, 1963; pp. 211–20.
For junior high school level; about two dozen assorted problems, including magic squares.

Steiger, A. A. von. Christmas puzzle. *M.T.* 60:848–49; Dec. 1967.

Steinen, R. F. More about 1965 and 1966. *M.T.* 59:737–38; Dec. 1966.

Sutcliffe, Alan. Waiting for a bus. *M.Mag.* 38:102–3; Mar.–Apr. 1965.

Trigg. C. W. Holiday greetings from thirty scrambled mathematicians. *S.S.M.* 54:679; Dec. 1954.

Chapter 11

Group Recreational Activities

11.1 Mathematics Clubs, Programs, Exhibits, Projects

"The fact is that there are few more 'popular' subjects than mathematics. Most people have some appreciation of mathematics, just as most people can enjoy a pleasant tune; and there are probably more people really interested in mathematics than in music. Appearances may suggest the contrary, but there are easy explanations."—G. H. Hardy, *A Mathematician's Apology*.

Anning, Norman. High school mathematics clubs. *M.T.* 26:70; Feb. 1933.
Bibliography.

Cameron, A. J. *Mathematical Enterprises for Schools.* New York: Pergamon Press, 1966. 188 pp.
Excellent source material for enrichment purposes.

Carnahan, Walter H. *Mathematics Clubs in High Schools.* Washington, D.C.: NCTM, 1958. 32 pp. (Pamphlet)

Cecilia, Sister Margaret. Mathematics projects. *M.T.* 54:527–30; Nov. 1961.
Gives suggestive list of 100 topics for projects or mathematics club programs.

Cordell, Christobel. *Dramatizing Mathematics.* Portland, Me.: J. Weston Walch, 1963. 170 pp.
Collection of 17 skits, contests, etc. suitable for math club and assembly programs.

Dalton, LeRoy C. Student-presented programs in a mathematics club. In *Chips from the Mathematical Log.* Mu Alpha Theta, 1966; p. 21.

Granito, D. What to do in a mathematics club. *M.T.* 57:35–40; Jan. 1964.

Hess, Adrien L. *Mathematics Projects Handbook.* Boston: D. C. Heath & Co., 1962. 60 pp. (Paper)
Bibliographic and source materials for typical projects and exhibits.

Johnson, Donovan. *Games for Learning Mathematics.* Portland, Me.: J. Weston Walch, 1963. 176 pp.
Description of 70 games, involving arithmetic, algebra, and geometry.

Johnson, Donovan. Panel games. *M.T.* 52:130–31; Feb. 1959.
Using "Twenty-Question" and "What's My Line?" techniques.

Johnson, Donovan, and Olander, C. E. *How to Use Your Bulletin Board.* Washington, D.C.: NCTM, 1955. 12 pp. (Pamphlet)

Kelly, Inez. The Garfield Mathematics Club. *M.T.* 45:37; Jan. 1952.

Levinson, R. XYZ Club. *High Points* 14:48; Nov. 1932.

Manheimer, Wallace. A club project in a modern use of mathematics. *M.T.* 50:350–55; May 1957.
Card tricks, Nim, and related recreations.

Mayall, R. N., and Mayall, M. L. *Sundials: How to Know, Use and Make Them.* Boston: C. T. Branford Co., 1958. 197 pp.

Olds, C. D. Miscellaneous topics for club talks. In *Chips from the Mathematical Log,* Mu Alpha Theta, 1966; p. 23.

Ransom, William R. *Thirty Projects for Mathematical Clubs and Exhibitions.* Portland, Me.: J. Weston Walch, 1961. 84 pp. (Paper)
A list of somewhat unusual topics.

Stephanie, Sister Mary. Activities for mathematics clubs. *Catholic School Journal* 56:214; Sept. 1956.

Todd, A. *The Maths Club.* 166 pp; 143 diagrams.
A handbook on mathematical activities suitable for 9–16 year old children.

Williams, A. J. List of mathematics clubs by states. *M.T.* 41:293; Oct. 1948.

11.2 Mathematical Contests and Competitions

Although there may be some controversy concerning the educational values of contests and competitions in mathematics, there is little doubt about their popularity. Indeed, the search for mathematical talent among young people has taken on a serious aspect in recent years, spurred on by the intrinsic needs of a scientific and technological society as well as by international rivalries, for better or for worse.

Berglass, Isidore, and Kalfas, Alfred. Math Fairs for New York State. *N. Y. State Math. Teachers Journal* 15:148–50; Oct. 1965.

Bissinger, Barnard. Science for a day. *M.T.* 57:546–51; Dec. 1964.

Bleicher, Michael N. Searching for mathematical talent in Wisconsin. *Am.M.Mo.* 72:412–16; Apr. 1965.

Buck, R. Creighton. A look at mathematical competitions. *Am.M.Mo.* 66:201–12; Mar. 1959.

Burkill, J. C., and Cundy, H. M. *Mathematical Scholarship Problems.* Cambridge University Press, 1961. 118 pp.

Charosh, Mannis, ed. *Mathematical Challenges.* Washington, D.C.: NCTM, 1965. 135 pp. (Paper)

A collection of problems for students in Grades 7 through 12.

Fagerstrom, William H., and Lloyd, Daniel B. The National High School Mathematics Contest. *M.T.* 51:434–39; Oct. 1958.

Gruver, Howell L. *School Mathematics Contests: A Report,* Washington, D.C.: NCTM, 1968. 41 pp. (Paper)

Discusses types of contests, examples of contests, procedures and practices, local and national contests, information about specific individual contests.

Hofeditz, P. H. Sequoia High School Math Team. *M.T.* 49:473; Oct. 1956.

Hungarian Problem Book I. *Eötvös Competitions,* 1894–1905. New York: Random House, L. W. Singer Co. 111 pp. (Paper)

Hungarian Problem Book II. *Eötvös Competitions,* 1906–1928. New York: Random House, L. W. Singer Co. 120 pp. (Paper)

Kneale, Brendan. A mathematics competition in California. *Am.M.Mo.* 73:1006–10; Nov. 1966.

Lenchner, George. Mathletes. *M.T.* 52:113–14; Feb. 1959.

The MAA Problem Book, II. *New Mathematical Library, no. 17.* New York: Random House, L. W. Singer Co., 1966. 112 pp. (Paper)

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"The play's the thing!"—*Hamlet.*

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Gives a list of 34 mathematics plays (without titles) that appeared in various issues of the *Mathematics Teacher* from 1924–54.

Cordell, Christobel. *Dramatizing Mathematics.* Portland, Me.: J. Weston Walch, 1963. 170 pp. (Paper)

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Gray, Mrs. J. T. Historically speaking. *S.S.M.* 52:345–56; 1952.

A play about the development of zero, weights, measures and time; suitable for intermediate and junior high school grades.

Schaaf, W. L. Mathematical plays and programs. *M.T.* 44:526–28; 1951.
Annotated list of 50 plays and pageants about mathematics.

Slaught, H. E. The evolution of numbers. A play. *Rec.M.M.*, no. 9, pp. 3–10; June 1962.
Reprint of a well-known play based on mathematical history.

Willerding, Margaret. Dramatizing mathematics. *S.S.M.* 60:99–104; 1960.
Extensive bibliography on mathematical plays.

Chapter 12

Mathematics and the Fine Arts

12.1 Aesthetics and Mathematics

"Art is an expression of the *world order* and is therefore orderly, organic, subject to mathematical law, and susceptible of mathematical analysis."—Claude Bragdon, *The Beautiful Necessity.*

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Hillman, T. P. Colors, geometric forms, art, and mathematics. *A.T.* 14:448–52; Oct. 1967.

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Scholarly discourse on the history of geometric ideas and the development of art.

Kandinsky, Wassily. *Point and Line to Plane.* 2d ed. New York: Solomon R. Guggenheim Foundation, 1947. 197 pp.

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Lietzmann. Walther. *Mathematik und bildende Kunst.* Breslau: F. Hirt, 1931. 150 pp.

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Morse, Marston. Mathematics and the arts. *Yale Review,* vol. 40, no. 4, pp. 604–12; June 1951.

Mott-Smith, John. Computers and art. *A.T.* 16:169–72; Mar. 1969.

Ogden, R. M. Naive geometry in the psychology of art. *American Journal of Psychology* 49:198–216; 1937.

Schillinger, Joseph. *The Mathematical Basis of the Arts.* New York: Philosophical Library, 1948. 696 pp.

A ponderous omnibus work presenting a unique approach to mathematical concepts in design, music and other arts; variously acclaimed as well as denigrated.

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12.2 Architecture and Mathematics

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Scholfield, P. H. *The Theory of Proportion in Architecture.* Cambridge University Press, 1958.

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Sykes, Mabel. *Source Book of Problems for Geometry.* Boston: Allyn & Bacon, 1912.

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Die Mathematik in der Ornament, pp. 57–77.

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Menninger, Karl. *Zwischen Raum und Zahl*. Frankfurt a. M.: Ullstein Bücher, 1960. 222 pp. (Paper)

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12.4 Painting—Drawing—Perspective

"For several reasons the problem of depicting the real world led the Renaissance painters to mathematics. . . . Stripped of color and substance, the objects that painters put on canvas are geometrical bodies located in space. The language for dealing with these idealized objects, the properties they possess as idealizations, and the exact relationships that describe their relative locations in space are all incorporated in Euclidean geometry. The artists need only avail themselves of it."—Morris Kline, *Mathematics in Western Culture.*

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12.5 The Golden Measure—Dynamic Symmetry

"The Golden Section therefore imposes itself whenever we want by a new subdivision to make two equal consecutive parts or segments fit into a geometric progression, combining thus the threefold effect of equipartition, succession, continuous proportion; the use of the Golden Section being only a particular case of a more general rule, the recurrence of the same proportions in the elements of a whole."—Heinrich Timerding.

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12.6 Literature and Mathematics

"Professional poets are very often, more frequently than mathematicians, bad critics of themselves and their work. The wisest of them refuse to talk

about poetry, leaving analysis and description to their more voluble companions, the critics."—Scott Buchanan, *Poetry and Mathematics.*

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12.7 Music and Mathematics

There is geometry in the humming of the strings.
There is music in the spacing of the spheres.

Pythagoras

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Harkin, Duncan. *Fundamental Mathematics.* New York: Prentice-Hall, 1941.

"Octaves; harmonics," p. 35; 76–78.

Helmholtz, Hermann F. L. *On the Sensations of Tone as a Physiological Basis for the Theory of Music.* (Trans. by A. J. Ellis, London, 1863.) New York: Dover Publications, 1954. 576 pp.

Hooke, Robert, and Shaffer, Douglas. *Math and Aftermath.* New York: Walker & Co., 1965.

"Of Clocks and Violins," pp. 68–77.

Jeans, Sir James. The Mathematics of Music. (In James R. Newman, *The World of Mathematics.* New York: Simon & Schuster, 1956; pp. 2278–2309.)

Kac, Mark. Can one hear the shape of a drum? *Am.M.Mo.,* vol. 73, no. 4, pt. II, pp. 1–23; Apr. 1966.

Highly technical discussion of the mathematics and physics of the properties of a stretched membrane under tension, as in a tambourine or drumhead.

Land, Frank. *The Language of Mathematics.* London: John Murray, 1960; Garden City, N.Y.: Doubleday & Co., 1963.

"Logs, Pianos and Spirals," pp. 117–32.

Langer, Susan K. A set of postulates for the logical structure of music. *Monist* 39:561–70; 1929.

Rather technical application of Boolean algebra.

Lawlis, Frank. The basis of music-mathematics. *M.T.* 60:593–96; Oct. 1967.

Meale, Elmer B. The two most original creations of the human spirit. *M.Mag.* 35:13–20; 1962.

Oh Empty Set! Oh Empty Set! What if Music and Math Never Met? *Music Educators Journal* 55:63–66; Sept. 1968.

Pines, Gladys. Mathematics in the arts (music). *Summation (ATM)*, vol. 11, no. 6, pp. 58–60; June 1966.

Rhoades, Patrick Alan. Pi in the key of C. *Indiana Mathematics News Letter*, Oct. 1966, pp. 3–4.

Rice, Jack A. The affinity of mathematics to music. *M.T.* 61:268–71; Mar. 1968.

Ridout, Theodore C. Sebastian and the "Wolf." *M.T.* 48:84–86; 1955.

Saminsky, Lazare. *Physics and Metaphysics of Music and Essays on the Philosophy of Mathematics.* The Hague: Martinus Nijhoff, 1957; 151 pp.

A curious work; sophisticated and scholarly, but unorganized and controversial.

School Mathematics Study Group. Reprint Series. Edited by W. L. Schaaf. *Mathematics and Music.* S.M.S.G., Stanford Univ., 1967. 25 pp. (Paper)

Essays by Elmer Mode, A. R. Amir-Moéz, and Theodore Ridout.

Silver, A. L. Leigh. Some musico-mathematical curiosities. *M.Gaz.* 48:1–17; Feb. 1964.

Tannery, Paul. *Mémoires Scientifiques.* Vol. 3. Paris: Gauthier-Villars, 1915.

Du rôle de la musique grecque dans le développement de la mathématique pure; pp. 68–89.
Sur un point d'histoire de la musique grecque; pp. 90–96.
Sur les intervalles de la musique grecque; p. 97–115.
Un traité grec d'arithmétique antérieur à Euclide; pp. 244–50.
Sur le Spondaïsme dans l'ancienne musique grecque; pp. 299–309.

Thernell, Beverly. Correlating music with mathematics. In *The Mathematics βa^r.* (Dept. of Math., Florida Atlantic University), vol. 2, no. 1, pp. 20–27; Mar. 1967.

Valens, Evans G. *The Number of Things: Pythagoras, Geometry, and Humming Strings.* New York: E. P. Dutton & Co., 1964.

Chapter 14: "Geometry for Listening"; Chapter 15: "Means and Ends"; Chapter 16: "Harmony and Harmonics"; Chapter 17: "Music of the Spheres."

12.8 Music and Computers

Apparently we have entered the age in which musical sounds, including the singing voice, can be generated by a suitable digital computer. To this end certain wave forms can be generated rather readily: sine waves, rectangular waves and sawtooth waves. It is too early, however, to assess the significance of this development.

Berkeley, Edmund. *The Computer Revolution.* Garden City, N.Y.: Doubleday & Co., 1962.

"Music by Computers," pp. 170–71.

Hiller, Lejaren. Computer music. *Sci.Am.*, Dec. 1959.

Hiller, Lejaren, and Beauchamp, James. Research in music with electronics. *Science* 150:161–69; Oct. 8, 1965.

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Mathews, M. V. An acoustic compiler for music and psychological stimuli. *Bell System Technical Journal* 40:677; 1961.

Mathews, M. V.; Pierce, J. R.; and Guttman, N. Musical sounds from digital computers. *Gravesaner Blaetter*, vol. 6, no. 23/24, p. 119; 1926.

Moles, A. A. *Les musiques expérimentales.* Paris: Editions du Cercle d'Art Contemporain, 1960.

Pierce, J. R., et al. Music from mathematics. *Science* 139:28–29; Jan. 4, 1963.

This is a review of a Decca recording (DL9103, 1962. 33⅓ rpm) of music played by IBM 7090 Computer and digital-to-sound transducer. Review by Melville Clark, Jr.

Pinkerton, Richard C. Information theory and melody. *Sci.Am.* 194:77–86; Feb. 1956.

Prieberg, F. K. *Musica ex Machina.* Berlin: Ullstein, 1960.

Strang, Gerald. The Computer in musical composition. *Computers & Automation*, vol. 15, no. 8, pp. 16–17; Aug. 1966.

Chapter 13

The Naturalist and Mathematics

13.1 The Naturalist and Mathematics

"There is no science which teaches the harmonies of nature more clearly than mathematics."—Paul Carus.

"The essential fact is that *all* the pictures which science now draws of nature, and which alone seem capable of according with observational facts, are *mathematical pictures.*"—Sir James Jeans.

Alfred, Brother. On the trail of the California Pine. *Fib.Q.* 6:69–76; Feb. 1968.
Discussion of spiral patterns on California pine cones.

"Animal Counting." *Nature*, 33:45.

Basin, S. L. The Fibonacci sequence as it appears in Nature. *Fib.Q.*, Feb. 1963, pp. 53–56.

Baugher, Carol. An adventure with spirals. *Pentagon* 20:78–85; Spring 1961.
Spirals in nature and in art.

Blake, Sue. Some of Nature's curves. *S.S.M.* 36:245–49, 486–89, 717–21; 1936.

Bowers, H., and Bowers, J. *Arithmetical Excursions.* New York: Dover Publications, 1961.
"Nature and Number," pp. 257–62.

Boys, C. V. *Soap Bubbles: Their Colours and the Forces which Mold Them.* New York: Dover Publications, 1959. 193 pp. (Paper)
Reprint of the 2nd edition of 1912.

Colman, Samuel, and Coan, C. Arthur. *Nature's Harmonic Unity.* New York: G. P. Putnam's Sons, 1912. 327 pp.

Cook, T. A. *The Curves of Life.* London; New York: Henry Holt & Co., 1914.

Coxeter, H. S. M. *Introduction to Geometry.* New York: John Wiley & Sons, 1961.
Phyllotaxis: pp. 169–72.

Dantzig, Tobias. *Number, the Language of Science.* New York: Macmillan Co., 1930.
Number sense in animals, pp. 1–4.

Emch, Arnold. Mathematics and engineering in Nature. *Popular Science* 79:450–58; 1911.

Fabre, Jean Henri. The Geometry of the Epeira's Web. In *Mathematics,* by Samuel Rapport and Helen Wright. New York: Washington Square Press, 1963; pp. 269–78.

Gardner, Martin. *The Ambidextrous Universe.* New York: Basic Books, 1964. 294 pp.

Discussion of mirror-imagery, asymmetry in plants and animals, the fourth dimension, and related topics.

Ginsburg, Abraham M. The sine function in Nature. *High Points* 19:69–70; 1937. (Also, reviewed in *M.T.* 31:35; 1938.)

Hickey, M. Efficiency of certain shapes in Nature and engineering. *M.T.* 32:129–33; 1939.

Insects that Recognize Numbers. *New York Times,* April 4, 1954 (Waldameer Kaempfert).

Koehler, O. Ability of birds to count. *Nature* 168:373–75; Sept. 1, 1951.

Kramer, Edna. *The Main Stream of Mathematics.* New York: Oxford University Press, 1951; Chapter 2, pp. 23–47.

Mackenzie, Fred. Geometry of Bermuda calcareous dune cross-bedding. *Science* 19:1449–50; June 19, 1964.

Patterns in Nature.

Macmillan, Hugh. The numerical relations of Nature. *London Quarterly Review,* n.s., 6:1–24; 1901.

McNabb, Sister Mary. Phyllotaxis. *Fib.Q.,* Dec. 1963, pp. 57–60.

Pettigrew, J. Bell. *Design in Nature.* 3 vol. 1908.

Sanders, S. T. Mathematics and Nature. *N.M.M.* 16:58; 1941.

Shepherd, W. Nature's wonders and mysteries. *Science Digest* 27:46; June 1950.

Skidell, Akiva. Letter on Pascal's triangle exemplified in biology. *A.T.* 10:517; Dec. 1963.

Smith, Cyril S. The shape of things. *Sci.Am.* 190:58–64; Jan. 1954.

Thompson, D'Arcy. *On Growth and Form.* Cambridge University Press, 1966. 346 pp.

Weiss, Paul. Beauty and the beast: life and the rule of order. *Sci.Mo.* 81:286–99; Dec. 1955.

Excellent photographs; geometry of living organisms.

Whyte, Lancelot L., ed. *Aspects of Form: A Symposium on Form in Nature and Art.* New York: Elsevier, 1968. 254 pp.

Reprint of the 1951 edition with a new preface.

13.2 Bees and Honeycombs

"In building her hexagonal cells, with their floors consisting of three lozenges, the bee solves with absolute precision the arduous problem of how

to achieve the maximum result at a minimum cost, a problem whose solution by man would demand a powerful mathematical mind."—Henri Fabre.

Ball, W. W. Rouse. *Mathematical Recreations and Essays.* 10th ed. New York: Macmillan Co., 1926.

Chapter VIII: "Bees and their Cells."

Bleicher, M. N., and Fejes Tóth, L. Two-dimensional honeycombs. *Am.M.Mo.* 72:969–73; Nov. 1965.

Bryant, Sophia. On the ideal geometrical form of natural cell-structure. *Proceedings, London Mathematical Society* 16:311–15; 1884–85.

Fejes Tóth, L. What the bees know and what they do not know. *Bulletin, American Mathematical Society* 70:468–81; July 1964.

Bibliography; 10 references.

Maclaurin, Colin. On the bases of the cells wherein the bees deposit their honey. *Philosophical Transactions of the Royal Society* 42:565–71; London, 1743.

Siemens, David F. Of bees and mathematicians. *M.T.* 60:758–61; Nov. 1967.

Siemens, David F. The mathematics of the honeycomb. *M.T.* 58:334–37; Apr. 1965. Bibliography.

Thatcher, R. The bee's cell. *Math.Tchg.*, no. 44, p. 7; Autumn 1968.

13.3 Snowflakes and Crystals

"Only one in a multitude of snow crystals is so nearly symmetrical as to show no obvious irregularities. But this usual imperfection of form is common to nearly all substances, especially when their crystals are comparatively large or rapidly produced, and may be attributed to fortuitous disturbances of one kind or another in the course of their growth."—W. A. Bentley and W. J. Humphreys, *Snow Crystals.*

Abrahams, H. J. The exhibit as a supplementary method; crystallography. *S.S.M.* 36:950–56; 1936.

Bentley, W. A., and Humphreys, W. J. *Snow Crystals.* New York: McGraw-Hill Book Co., 1931; Dover Publications, 1962. 227 pp.

Over 200 pages of photographs, plus brief text and bibliography.

Bond, W. L. Mathematics of the physical properties of crystals. *Bell System Technical Journal* 22:1–72; Jan. 1943.

Buerger, M. J. *Elementary Crystallography: An Introduction to the Fundamental Geometrical Features of Crystals.* 1956.

Excellent discussion of symmetry, comprehensible in terms of simple mathematics.

Burckhardt, Johann J. *Die Bewegungsgruppen der Kristallographie.* Basel: Birkhauser Verlag, 1966. 209 pp.

Butler, G. Montague. *A Manual of Geometrical Crystallography*. New York: John Wiley & Sons.

Himmel, Albert. A classroom demonstration of the crystal systems. *S.S.M.* 65:56–61; Jan. 1965.

Holden, Allen, and Singer, Phyllis. *Crystals and Crystal Growing*. Garden City, N.Y.: Doubleday & Co., Anchor Books, 1960.

Nakaya, Ukchiro. *Snow Crystals; Natural and Artificial*. Cambridge: Harvard University Press, 1954. 510 pp.

Phillips, J. McKeeby. Motivating the study of solid geometry through the use of mineral crystals. *S.S.M.* 52:743–48; Dec. 1952. Also, *S.S.M.* 53:134–38; Feb. 1953.

Ridout, Theodore. The mathematical behavior of minerals. *M.T.* 42:88–90; 1949.

Rogers, A. F. Mathematical study of crystal symmetry. *Proceedings, American Academy of Arts and Sciences*, vol. 61, no. 7, pp. 161–203; June 1926.

Runnels, L. K. Ice. *Sci.Am.* 215:118–26; Dec. 1966.

Geometric structure of water molecules in ice crystals.

Tutton, A. E. H. *Crystallography and Practical Crystal Measurement*. 2d ed. 2 vols. 1922. Reprint. New York: Stechert-Hafner Service Agency, 1964.

An elaborate treatise.

Wade, F., and Mattox. *Elements of Crystallography*. New York: Harper & Bros., 1960.

Williams, Robert E. Space-filling polyhedron: its relation to aggregates of soap bubbles, plant cells, and metal crystallites. *Science* 161:276–77; July 19, 1968.

Highly technical discussion of two fourteen-faced space-filling polyhedrons.

Yale, Paul B. *Geometry and Symmetry*. San Francisco: Holden-Day, 1968. 288 pp.

Introduction to Euclidean, affine, and projective spaces with emphasis on symmetry; contains a chapter on crystallography. For mature readers.

Appendix

List of General Works on Mathematical Recreations

The following list of titles is limited to books, monographs, and pamphlets devoted to *general collections* of mathematical recreations or to miscellaneous recreational matters. Books and monographs dealing solely with a specific subject (e.g., "magic squares," or "cryptography," or "paperfolding," or "mazes," or "probability") have been deliberately excluded since they are already listed under an appropriate heading in the Bibliography proper.

Most of the items listed here were published after 1960. However, a number of earlier publications not listed in Volume I have been added; also, a few entries listed in the Supplement of Volume I have been repeated here for the reader's possible convenience.

Abraham, R. M. *Easy-to-Do Entertainments and Diversions with Cards, String, Coins, Paper and Matches.* New York: Dover Publications, 1961. 186 pp. (Paper)

Unabridged republication of the first edition (1932) of "Winter Nights' Entertainments."

Adams, John Paul. *Puzzles for Everybody.* New York: Avon Publications, 1955.

Adams, John Paul. *I'll Dare You To Solve This!* New York: Brekley Publishing Co., 1957. 123 pp.

Some conventional, many original puzzles.

Adams, Morley. *The Children's Puzzle Book.* London: Faber Popular Books, 1942. 80 pp.

Adler, Irving. *Magic House of Numbers.* New York: New American Library, 1957. 123 pp. (Paper)

Reprint of a work originally published by John Day, 1957.

Ahrens, Wilhelm. *Mathematische Spiele. Encyklopädie der Mathematischen Wissenschaften.* Vol. I, 1902; 96 pp.

Anderson, Robert Perry. *Mathematical Bingo.* Portland, Me.: J. Weston Walch, 1963. 75 + 21 pp.

Entertaining form of practice exercises, similar to conventional Bingo. Covers range of junior and senior high school mathematics.

Bachet, Claude-Gaspar de Méziriac. *Problèmes plaisantes et délectables qui se font par les nombres.* Paris: Albert Blanchard, 1959. 242 pp. (Paper)

New edition of a classic work.

Bakst, Aaron. *Mathematical Puzzles and Pastimes.* 2d ed. New York: D. Van Nostrand Co., 1965. 206 pp.

Barnard, Douglas St. Paul. *Adventures in Mathematics.* New York: Hawthorne Books, 1965. 130 pp.

Recreational mathematics for the general reader.

Barnard, Douglas St. Paul. *A Book of Mathematical and Reasoning Problems.* New York: D. Van Nostrand Co., 1963. 109 pp.

A collection of 50 problems taken from the author's "Brain Twister" column that appeared in the London *Observer.*

Barnard, Douglas St. Paul. *50 "Observer" Braintwisters.* London: Faber & Faber, 1962. 109 pp.

Barr, Stephen. *A Miscellany of Puzzles: Mathematical and Otherwise.* New York: Thomas Y. Crowell Co., 1965. 164 pp.

Bell, R. C. *Board and Table Games from Many Civilizations.* New York: Oxford University Press, 1960. 232 pp.

Boon, F. C. *Puzzle Papers in Arithmetic.* Revised ed. London: G. Bell & Sons, 1960.

Boyer, John; Strohm, Rufus; and Pryor, George. *Real Puzzles.* Baltimore: Norman Remington Co., 1925.

Brandes, Louis G. *4th Math. Wizard.* Portland, Me.: J. Weston Walch; 1962. 252 pp. (Paper)

Brooke, Maxey. *Fun for the Money.* New York: Charles Scribner's Sons, 1964. 96 pp.

A collection of 60 braintwisters, puzzles, and games with coins; some old, some new; some easy, some hard; all interesting.

Burger, Dionys. *Sphereland.* New York: Thomas Y. Crowell Co., 1965–66. 208 pp.

A witty and delightful story dealing with four domains: the straight world, congruence and symmetry, curved worlds, and expanding worlds. A worthy successor to Abbott's "Flatland."

Cadwell, J. S. *Topics in Recreational Mathematics.* Cambridge University Press, 1966. 176 pp.

Largely a collection of well-known items concerning π, prime numbers, dissections, polyhedra, map-coloring, etc. A few new items are presented: sequences of nested polygons; application of Fibonacci numbers to computer sorting; shapes for drilling square or triangular holes; and some others.

Callandreau, E. *Célèbres Problèmes Mathématiques.* Paris, 1949. 474 pp.

Carroll, Lewis (C. L. Dodgson). *Mathematical Recreations of Lewis Carroll.* Vol. 1: "*Symbolic Logic*" and "*The Game of Logic.*" (2 books bound as 1.) New York: Dover Publications, 1958. 199 + 69 pp.

The first book consists of some 400 logical problems involving syllogisms and sorites.

Carroll, Lewis (C. L. Dodgson). *Mathematical Recreations of Lewis Carroll.* Vol. 2: "*Pillow Problems*" and "*A Tangled Tale.*" (2 books bound as 1.) New York: Dover Publications, 1958. 109 + 152 pp.

Pillow Problems is a classical collection of 72 sophisticated "brainteasers."

Collingwood, Stuart Dodgson, ed. *Diversions and Digressions of Lewis Carroll.* (Formerly titled: *The Lewis Carroll Picture Book.*) New York: Dover Publications, 1961. 375 pp.

Chapter 5: "Curiosa Mathematica"; Chapter 6: "Games and Puzzles."

Davidson, Jessica, and Martin, William. *Mind in a Maze.* New York: Prentice-Hall, 1969.

For teenagers.

Delens, P. *Problèmes d'Arithmétique Amusante.* 4th ed. Paris: Vuibert, 1948. 164 pp. (Paper)

New edition of a classic work.

Derry, L. *Games and Number.* Studio Vista, 1965.

Dineeman, Howard P. *Superior Mathematical Puzzles.* London: George Allen & Unwin, 1968. 122 pp.

Disney, Walt. *Donald in Mathmagic Land.* (No. 1051). New York: Dell Publishing Co., 1959. 32 pp. (Paper)

Domoryad, A. P. *Mathematical Games and Pastimes.* (Trans. from the Russian by Halina Moss.) London: Pergamon Press, 1964. 288 pp.

A refreshing approach to "standard" mathematical recreations; well organized, sophisticated, and very readable.

Dudeney, Henry E. *Amusements in Mathematics.* New York: Dover Publications, 1958. 258 pp. (Paper)

A reprint of the original edition of 1917.

Dudeney, Henry E. *Amusements in Mathematics, with Solutions.* London: Nelson, 1949.

Dudeney, Henry E. *Canterbury Puzzles, with Solutions.* 7th ed. London: Nelson, 1949.

Dudeney, Henry E. *A Puzzle Mine. Puzzles Collected from Dudeney's Works.* by J. Tregoss. London, n.d.

Dudeney, Henry E. *536 Puzzles and Curious Problems.* Edited by Martin Gardner. New York: Charles Scribner's Sons, 1967. 329 pp.

A combination of most of the problems in two of Dudeney's most popular works: "Modern Puzzles" and "Puzzles and Curious Problems"; brief biographical notes on Dudeney and Sam Loyd.

Dumas, Enoch. *Arithmetic Games*. San Francisco, Calif.: Fearon Publishers, 1960. 56 pp. (Paper)

Dunn, Angela. *Mathematical Bafflers*. New York: McGraw-Hill Book Co., 1964. 217 pp.

Over 150 sophisticated problems, involving algebra, geometry, Diophantine equations, probability, logic, theory of numbers.

Dynkin, E. B., and Uspenskii, V. A. *Mathematical Conversations*. Gostekhizdat, Moscow, 1952. 200 pp. German translation. Berlin, 1956.

Some 200 problems in number theory, topology, probability, etc., together with their solutions.

Ellegon, C. *Mathematische Spielereien*. Vienna, 1865.

Emmet, E. R. *Brain Puzzler's Delight*. New York: Emerson Books, 1960. 256 pp.

Fadiman, Clifton. *Fantasia Mathematica*. New York: Simon & Schuster, 1958. 290 pp.

A collection of humorous stories and diversions related to mathematics.

Fadiman, Clifton. *The Mathematical Magpie*. New York: Simon & Schuster, 1962. 300 pp.

A delightful collection of humor about mathematics: aphorisms, apologues, anecdotes, poems, limericks, cartoons, essays, and curious.

Fallstrom, Edward. *Games Indoor and Outdoor and How to Play Them*. New York: Dover Publications, 1961. 366 pp. (Paper)

Fourmi. *Les Récréations Intelligentes*. 1881.

Fournier, S. *Les Nombres Premiers*. Paris, 1947. 159 pp.

Filipiak, Anthony S. *Mathematical Puzzles and Other Brain Twisters*. New York: A. S. Barnes & Co., 1960. 120 pp.

Originally published in 1942 under the title: *100 Puzzles: How to Make and How to Solve Them*.

Friedman, Helen Hill. *Put on Your Thinking Cap*. New York: Harbert-Schuman, 1969.

About 100 puzzles for ages 9 to 15.

Friend, J. Newton. *Man Numbers: Fun and Facts*. New York: Charles Scribner's Sons, 1961. 301 pp.

Fuchshtein, Jack. *Mathematical Fun, Games and Puzzles*. New York: Dover Publications, 1962. 300 pp. (Paper)

Considerable material dealing with percentage, business arithmetic, measurement, etc.

Funk, Julia S. *Puzzles and Graphs*. Washington, D.C.: NCTM, 1969. 72 pp. (Paper)

Gamow, George, and Stern, Marvin. *Puzzle-Math*. New York: Viking Press, 1958. 138 pp.

Many delightful puzzles designed to be read and thought.

Gardner, Martin. *Mathematical Puzzles.* New York: Thomas Y. Crowell Co., 1961.

For school children.

Gardner, Martin, ed. *Mathematical Puzzles of Sam Loyd.* New York: Dover Publications, 1959. 167 pp. (Paper)

More than 100 puzzles from Loyd's famous *Cyclopedia of Puzzles.*

Gardner, Martin, ed. *Mathematical Puzzles of Sam Loyd.* Vol. 2. New York: Dover Publications, 1960. 175 pp. (Paper)

Companion volume to the above.

Gardner, Martin. *New Mathematical Diversions from Scientific American.* Simon & Schuster, 1966. 253 pp.

The third of a series of companion volumes, each more interesting than its predecessor.

Gardner, Martin. *The Numerology of Dr. Matrix.* New York: Simon & Schuster, 1967. 112 pp.

Gardner, Martin. *Perplexing Puzzles and Tantalizing Teasers.* New York: Simon & Schuster, 1969.

For young folks.

Gardner, Martin. *The Scientific American Book of Mathematical Puzzles and Diversions.* New York: Simon & Schuster, 1959. 178 pp.

Sophisticated essays on mathematical recreations, with considerable new material.

Gardner, Martin. *The Second Scientific American Book of Mathematical Puzzles and Diversions.* New York: Simon & Schuster, 1961. 251 pp.

A companion volume to the above; many new diversions, such as tetraflexagons, Soma cubes, topology, Origami, etc.

Gardner, Martin. *The Unexpected Hanging and Other Mathematical Diversions.* New York: Simon & Schuster, 1969. 255 pp.

A collection of 20 essays on recreational mathematics.

Ghersi, I. *Matematica Dillettevole e. Curiosa.* Milan: Hoepli, 1963. 776 pp. (A reprint of the 4th edition.)

Goodman, A. W. *The Pleasures of Math.* New York: Macmillan Co., 1965. 224 pp.

Includes a number of challenging games.

Graham, L. A. *Ingenious Mathematical Problems and Methods.* New York: Dover Publications, 1959. 237 pp. (Paper)

Collection of 100 puzzles contributed by scores of mathematicians to an industrial magazine over a period of 18 years.

Graham, L. A. *The Surprise Attack in Mathematical Problems.* New York: Dover Publications, 1967. 125 pp. (Paper)

Problems dealing with measurement of geometrical spaces, probabilities, distances, number systems other than the decimal, interesting number rela-

tions, relative motion, and the discovery of the simplest solution through purely logical means.

Greenblatt, M. H. *Mathematical Entertainments.* New York: Thomas Y. Crowell Co., 1965; London: Allen & Unwin, 1968. 160 pp.

Haber, Philip. *Peter Pauper's Puzzles and Posers.* Mt. Vernon, N.Y.: Peter Pauper Press, 1963. 62 pp.

Heafford, Philip. *The Math Entertainer.* London: Hutchinson & Co.; New York: Emerson Books, 1959. 176 pp.

Heafford, Philip. *Mathematics for Fun: A Quiz Book.* London: Hutchinson & Company, 1957. 176 pp.

Fifty short quizzes dealing with mathematical miscellania, including historical material.

Heafford, Philip, and Babb, Henry. *A New Mathematics for Fun: A Quiz Book.* London: Hutchinson & Co., 1966. 180 pp. (Paper)

Hoffmann, Prof. (A. J. Lewis). *Puzzles Old and New.* London: F. Warne & Co., 1893. 394 pp.

Horne, Sylvia. *Patterns and Puzzles in Mathematics.* Franklin Publications, Inc., c/o A. C. Vroman, Inc., 2085 East Foothill Blvd., Pasadena, California 91109; 1968.

For intermediate grade pupils.

Hunter, J. A. H. *Figurets: More Fun with Figures.* New York: Oxford University Press, 1958. 116 pp.

Hunter, J. A. H. *Fun with Figures.* New York: Dover Publications, 1965. 109 pp. (Paper)

Hunter, J. A. H. *Hunter's Math Brain Teasers.* New York: Bantam Books, 1965. 147 pp. (Paper)

Hunter, J. A. H. *More Fun with Figures.* New York: Dover Publications, 1966. 116 pp. (Paper)

Hunter, J. A. H., and Madachy, Joseph S. *Mathematical Diversions.* New York: D. Van Nostrand Co., 1963. 178 pp.

Hutton, Charles. *Recreations in Mathematics and Natural Philosophy.* 4 vols. London, 1803.

Jacoby, Oswald, and Benson, William H. *Mathematics for Pleasure.* London: Victor Gollancz, 1962; New York: McGraw-Hill Book Co., 1962. 191 pp.

Johnson, Donovan, and Glenn, William H. *Fun with Mathematics.* St. Louis, Mo.: Webster Publishing Co., 1960. 43 pp. (Pamphlet)

Kaplan, P. *Posers: 80 Delightful Hurdles for Reasonably Agile Minds.* New York: Harper, 1963. 86 pp.

Kempner, A. J. *Paradoxes and Common Sense.* New York: D. Van Nostrand Co., 1959. 22 pp. (Pamphlet)

Kendall, P. M. H., and Thomas, G. M. *Mathematical Puzzles for the Connoisseur.* London: Charles Griffin & Co., 1962. 161 pp.

Kendall, P. M. H., and Thomas, G. M. *Mathematical Puzzles for the Connoisseur.* New York: Thomas Y. Crowell Co., 1964. 161 pp.

More than 100 brainteasers, interestingly presented.

Kettlecamp, Larry. *Puzzle Patterns.* New York: William Morrow & Co., 1963.

Geometric puzzles, ciphers, anagrams, etc., for children.

King, Tom. *The Best 100 Puzzles, Solved and Answered.* London: W. Foulsham & Co., n.d.; 62 pp. (Paper)

Klein, Felix, et al. *Famous Problems and Other Monographs.* New York: Chelsea Publishing Co., 1962.

Contains a reprint of F. Klein's classic little book.

Knott, C. G., trans. *Tom Tit: Scientific Amusements.* 1918. 413 pp. (Scarce)

Langman, Harry. *Play Mathematics.* New York: Hafner Publishing Co., 1962. 216 pp.

A fresh approach to mathematical recreations.

Lasker, Edward. *Go and Go-Moku.* New York: Dover Publications, 1960. 215 pp.

Lausman, Raymond. *Fun With Figures.* New York: McGraw-Hill Book Co., 1965. 245 pp.

Lehmann, M. B. *Neue Mathematische Spiele.* 5th ed. Wiesbaden, 1932.

Lieber, H. G., and Lieber, L. R. *Modern Mathematics for T. C. Mits.* New York: W. W. Norton & Co., 1944; London: Allen & Unwin, 1946.

Lietzmann, Walther. *Lustiges und Merkwürdiges von Zahlen und Formen.* Breslau: F. Hirt, 1922. 8th ed. Göttingen: Vandenhoeck and Ruprecht, 1955.

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Over 100 problems and puzzles, many of them new.

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Merrill, Helen. *Mathematical Excursions.* New York: Dover Publications, 1958. 145 pp. (Paper)

Collection of recreational problems, magic squares, short cuts, etc.; for high school students.

Meyer, Jerome S. *Fun With Mathematics.* New York: Fawcett World Library, 1957. 176 pp. (Paper)

Reprint of an earlier edition; suitable for high school pupils.

Meyer, Jerome, and Hanlon, Stuart. *Fun With the New Math.* New York: Hawthorn Books, 1967. 120 pp.

Nødde-Knaekkeren. Copenhagen: Politikens Forlag, 1955. 232 pp. (Paper)

Conventional collection of mathematical recreations.

O'Beirne, T. H. *Puzzles and Paradoxes.* New York: Oxford University Press, 1965. 238 pp.

A collection of puzzles in which the solutions form an integral part of the text.

Ogilvy, C. Stanley. *Through the Mathescope.* New York: Oxford University Press, 1956. 162 pp.

Miscellaneous topics: algebraic oddities, probability, pi and pi-makers, topology, etc.

Ogilvy, C. Stanley. *Tomorrow's Math.* New York: Oxford University Press, 1962. 182 pp.

Over 150 problems related to topology, probability and combinatorics, game theory, etc.

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Phillips, Hubert [Caliban]. *My Best Puzzles in Logic and Reasoning.* New York: Dover Publications, 1961. 107 pp.

An excellent collection of "logic problems," almost all original.

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Some 300 problems, mostly new.

Phillips, Hubert. *The Sphinx Problem Book.* London: Faber & Faber, 1934. 207 pp.

One hundred intriguing puzzles.

Phillips, Hubert, Shovelton, S. T., and Marshall, C. S. *Caliban's Problem Book: Mathematical, Inferential and Cryptographic Puzzles.* New York: Dover Publications, 1961. 180 pp. (Paper)

A republication of the well-known book which first appeared in 1933.

Ransom, William R. *Pastimes with String and Paper.* North Quincy, Mass.: Christopher Publishing Co., 1963.

Ranucci, Ernest. *Four by Four.* Boston: Houghton Mifflin Co., 1968. 60 pp.

An assortment of recreations using a 4×4 network of squares; appropriate for secondary school students.

Reinfeld, Don, and Rice, David. *101 Mathematical Puzzles and How to Solve Them.* New York: Sterling Publishing Co., 1960; Paperback edition, New York: Cornerstone Library, 1960. 123 pp.

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Sawyer, W. W. *Mathematician's Delight.* New York: Penguin Books, 1946. 215 pp.

Not strictly "recreational," but stimulating and interesting.

Scarne, John. *Scarne on Teeko.* New York: Crown Publishers, 1955. 256 pp.

Teeko: a game played on a 5×5 board.

School Mathematics Study Group. *Puzzle Problems and Games Project: Final Report.* Studies in Mathematics, vol. 18. S.M.S.G., 1968. 218 pp.

Concerned with Nim-type games, polyominoes, symmetry, counting machines, games with addition and multiplication tables, geometric puzzles, etc.

Schubert, Hermann. *Mathematische Mussestunden.* 3d ed. 3 vols. Leipzig: Göschen, 1907–09. 12th ed., rev. and abridged, 1 vol. Berlin: W. de Gruyter, 1964.

Schubert, Hermann. *Mathematische Mussestunden: Eine Sammlung von Geduldspielen, Kunststücken und Unterhaltungsaufgaben mathematischer Natur.* (New edition, edited by Joachim Erlebach). 1967, 263 pp.

Schubert, Hermann, and Fitting, F. *Mathematische Mussestunden.* 11th ed. Berlin: Walter de Gruyter, 1953. 271 pp.

Conventional collection of mathematical recreations, but ever popular.

Schuh, Fred. *The Master Book of Mathematical Recreations.* (Trans. by F. Gobel; edited by T. H. O'Beirne.) New York: Dover Publications, 1968. 430 pp. (Paper)

A classic; comprehensive and scholarly.

Seuss, Dr., and Geisel, Theodor Seuss. *Pocket Book of Boners.* New York: Pocket Books, 1931.

Seymour, Dale, and Gidley, Richard. *Eureka!* 1850 Fulton St., Palo Alto, Calif.: Creative Publications, 1967, 1968. 128 pp. (Paper)

A refreshing collection of assorted mathematical puzzles, designs, and curiosities.

Simon, William. *Mathematical Magic.* New York: Charles Scribner's Sons, 1964. 187 pp.

Number tricks; calendar tricks; tricks with playing cards and other ordinary objects; mental calculation; magic squares; topological tricks.

Skotte, Ray, and Magnuson, Yngve. *Math Fun.* Minneapolis, Minn.: The Authors, 6380 Monroe Street, Minneapolis 21; 1959. 88 pp. (Paper)

Smith, Eugene P. Some puzzlers for thinkers. *NCTM Twenty-seventh Yearbook,* 1963; pp. 211–20.

A collection of twenty-odd stock puzzles, with a brief note on magic squares.

Sperling, Walter. *Auf du und du mit Zahlen.* Zurich: Albert Müller Verlag, 1955. 114 pp.

Number curiosities, number games, and short cuts in computation.

Sperling, Walter. *Spiel und Spass fürs ganze Jahr.* Zurich: Albert Müller Verlag. 1951.

Spitzer, Herbert F. *Enrichment of Arithmetic.* New York: McGraw-Hill Book Co., Webster Div., 1964. 576 pp.

Collection of number games, puzzles and recreations; Grades 1 through 8; exercises for exploration and discovery.

Sprague, R. *Unterhaltsame Mathematik: Neue Probleme, Überraschende Lösungen.* Brunswick: Friedrich Vieweg & Sohn, 1961. 51 pp. (Paper)

Sprague, R. *Recreation in Mathematics.* Translated by T. O'Beirne. London: Blackie, 1963. 60 pp. (Paper)

Brief collection of 30 original mathematical puzzles.

Steinhaus, Hugo. *Mathematical Snapshots.* New York: Oxford University Press. (Third American edition). 1969. 311 pp.

Steinhaus, Hugo. *One Hundred Problems in Elementary Mathematics.* New York: Basic Books, 1963.

A collection of 113 unusual brain-crackers, most of them brand new.

Straszewicz, S. *Mathematical Problems and Puzzles from the Polish Mathematical Olympiads.* New York: Pergamon Press, 1965. 367 pp.

Summers, George J. *Fifty Problems in Deduction.* New York: Dover Publications, 1969. (Paper)

Summers, George J. *New Puzzles in Logical Deduction.* New York: Dover Publications, 1968. 121 pp. (Paper)

Synge, J. L. *Science: Sense and Nonsense.* London: Cape, 1951.

Tietze, Heinrich. *Famous Problems of Mathematics.* New York: Graylock Press, 1965. 367 pp.

Tocquet, Robert. *Magic of Numbers.* New York: Wehman Bros., 1960. 160 pp.

Arithmetical tricks and short cuts; mental calculation; calculating prodigies.

Tocquet, Robert. *Magic House of Numbers.* New York: Fawcett World Library, Premier Books, 1961. (Paper)

Number tricks, rapid calculation, etc.

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Contains 270 problems, with solutions, ranging from relatively simple to definitely sophisticated problems; covers a wide variety of material, including Fibonacci series, Schlegel diagrams, square triangular numbers, and other topics; not classified as to content or difficulty.

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Webster, D. *Brain Boosters.* New York: Natural History Press, 1966.

Wilkens, J. *Mathematical Magick, or the Wonders that may be performed by Mechanicall Geometry.* Reprint of the 1648 edition. London: 1968; 295 pp.

Williams, W. T., and Savage, G. H. *The Strand Problems Book: A Modern Anthology of Perplexities and Tantalizers.* London: George Newnes, n.d. 159 pp.

One hundred inferential and mathematical problems.

Wyatt, E. M. *Puzzles in Wood.* 10th Printing. Milwaukee: Bruce Publishing Co., 1956.

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